

Deduction

$$\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = \log_e e = 1$$

We know that

$$\lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \log_a a$$

Put $a = e$

$$\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = \log_e e = 1$$

Important results to remember

- (i) $\lim_{x \rightarrow +\infty} (e^x) = \infty$
- (ii) $\lim_{x \rightarrow -\infty} (e^x) = \lim_{x \rightarrow -\infty} \left(\frac{1}{e^{-x}} \right) = 0$
- (iii) $\lim_{x \rightarrow \pm\infty} \left(\frac{a}{x} \right) = 0$, where a is any real number.

EXERCISE 1.3

Q.1 Evaluate each limit by using theorems of limits.

- | | |
|--|--|
| (i) $\lim_{x \rightarrow 3} (2x + 4)$ | (ii) $\lim_{x \rightarrow 1} (3x^2 - 2x + 4)$ |
| (iii) $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4}$ | (iv) $\lim_{x \rightarrow 2} x \sqrt{x^2 - 4}$ |
| (v) $\lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5})$ | (vi) $\lim_{x \rightarrow 2} \frac{2x^3 + 5x}{3x - 2}$ |

Solution:

- (i) $\lim_{x \rightarrow 3} (2x + 4) = \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (4)$
 $= 2 \lim_{x \rightarrow 3} x + 4$
 $= 2(3) + 4 = 6 + 4 = 10$ Ans.
- (ii) $\lim_{x \rightarrow 1} (3x^2 - 2x + 4) = \lim_{x \rightarrow 1} (3x^2) - \lim_{x \rightarrow 1} (2x) + \lim_{x \rightarrow 1} (4)$
 $= 3 \lim_{x \rightarrow 1} x^2 - 2 \lim_{x \rightarrow 1} x + 4$
 $= 3(1)^2 - 2(1) + 4$
 $= 3 - 2 + 4$
 $= 5$ Ans.
- (iii) $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4} = [\lim_{x \rightarrow 3} (x^2 + x + 4)]^{1/2}$

$$\begin{aligned}
 &= [\lim_{x \rightarrow 3} x^2 + \lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 4]^{1/2} \\
 &= (3^2 + 3 + 4)^{1/2} \\
 &= (9 + 7)^{1/2} = (16)^{1/2} = (4^2)^{1/2} = 4 \text{ Ans.}
 \end{aligned}$$

(iv) $\lim_{x \rightarrow 2} x\sqrt{x^2 - 4} = [\lim_{x \rightarrow 2} (x)] [\lim_{x \rightarrow 2} (x^2 - 4)^{1/2}]$

$$\begin{aligned}
 &= 2 [\lim_{x \rightarrow 2} (x^2 - 4)]^{1/2} \\
 &= 2 [\lim_{x \rightarrow 2} x^2 - \lim_{x \rightarrow 2} 4]^{1/2} \\
 &= 2 (4 - 4)^{1/2} \\
 &= 2(0)^{1/2} \\
 &= 0 \text{ Ans.}
 \end{aligned}$$

(v) $\lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5}) = \lim_{x \rightarrow 2} (x^3 + 1)^{1/2} - \lim_{x \rightarrow 2} (x^2 + 5)^{1/2}$

$$\begin{aligned}
 &= [\lim_{x \rightarrow 2} (x^3 + 1)]^{1/2} - [\lim_{x \rightarrow 2} (x^2 + 5)]^{1/2} \\
 &= [\lim_{x \rightarrow 2} x^3 + \lim_{x \rightarrow 2} 1]^{1/2} - [\lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 5]^{1/2} \\
 &= (8 + 1)^{1/2} - (4 + 5)^{1/2} \\
 &= (9)^{1/2} - (9)^{1/2} = (3^2)^{1/2} - (3^2)^{1/2} = 3 - 3 \\
 &= 0 \text{ Ans.}
 \end{aligned}$$

(vi) $\lim_{x \rightarrow -2} \frac{2x^3 + 5x}{3x - 2} = \frac{\lim_{x \rightarrow -2} (2x^3 + 5x)}{\lim_{x \rightarrow -2} (3x - 2)}$

$$\begin{aligned}
 &= \frac{2 \lim_{x \rightarrow -2} x^3 + 5 \lim_{x \rightarrow -2} x}{3 \lim_{x \rightarrow -2} x - \lim_{x \rightarrow -2} 2} \\
 &= \frac{2(-2)^3 + 5(-2)}{3(-2) - 2} \\
 &= \frac{2(-8) - 10}{-6 - 2} = \frac{-16 - 10}{-8} = \frac{-26}{-8} = \frac{13}{4} \text{ Ans}
 \end{aligned}$$

Q.2 Evaluate each limit by using algebraic techniques.

(i) $\lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1}$

(ii) $\lim_{x \rightarrow 1} \left(\frac{3x^3 + 4x}{x^2 + x} \right)$

(iii) $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 + x - 6}$

(iv) $\lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x}$ (Lhr. Board 2009)

(v) $\lim_{x \rightarrow -1} \left(\frac{x^3 + x^2}{x^2 - 1} \right)$

(vi) $\lim_{x \rightarrow 4} \frac{2x^2 - 32}{x^3 - 4x^2}$

- (vii) $\lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$ (Lhr. Board 2006)
- (viii) $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ (Lhr. Board 2004)
- (ix) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m}$

Solution:

(i) $\lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1}$ $\left(\frac{0}{0}\right)$ form

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1} &= \lim_{x \rightarrow -1} \frac{x(x^2 - 1)}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{x(x + 1)(x - 1)}{x + 1} \\ &= \lim_{x \rightarrow -1} x(x - 1) \\ &= -1(-1 - 1) \\ &= -1(-2) = 2 \quad \text{Ans.} \end{aligned}$$

(ii) $\lim_{x \rightarrow 1} \left(\frac{3x^3 + 4x}{x^2 + x} \right) = \frac{3(1)^3 + 4(1)}{(1)^2 + 1}$

$$= \frac{3 + 4}{2} = \frac{7}{2} \quad \text{Ans.}$$

(iii) $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 + x - 6}$ $\left(\frac{0}{0}\right)$ form (Gujranwala 2007, Lahore Board 2008)

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 + x - 6} &= \lim_{x \rightarrow 2} \frac{(x)^3 - (2)^3}{x^2 + 3x - 2x - 6} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2x + 4)}{x(x + 3) - 2(x + 3)} \quad [\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)] \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2x + 4)}{(x + 3)(x - 2)} \\ &= \frac{(2)^2 + 2(2) + 4}{2 + 3} = \frac{4 + 4 + 4}{5} = \frac{12}{5} \quad \text{Ans.} \end{aligned}$$

(iv) $\lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x}$ $\left(\frac{0}{0}\right)$ form (Lahore Board 2009)

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x} &= \lim_{x \rightarrow 1} \frac{(x - 1)^3}{x(x^2 - 1)} \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)^3}{x(x + 1)(x - 1)} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 1} \frac{(x-1)^2}{x(x+1)} \\
 &= \frac{(1-1)^2}{1(1+1)} = \frac{0}{2} = 0 \quad \text{Ans.}
 \end{aligned}$$

(v) $\lim_{x \rightarrow -1} \left(\frac{x^3 + x^2}{x^2 - 1} \right) \left(\frac{0}{0} \right) \text{ form}$

$$\begin{aligned}
 \lim_{x \rightarrow -1} \left(\frac{x^3 + x^2}{x^2 - 1} \right) &= \lim_{x \rightarrow -1} \frac{x^2(x+1)}{(x+1)(x-1)} \\
 &= \lim_{x \rightarrow -1} \frac{x^2}{x-1} \\
 &= \frac{(-1)^2}{-1-1} = \frac{-1}{2} \quad \text{Ans.}
 \end{aligned}$$

(vi) $\lim_{x \rightarrow 4} \frac{2x^2 - 32}{x^3 - 4x^2} \left(\frac{0}{0} \right) \text{ form}$

$$\begin{aligned}
 \lim_{x \rightarrow 4} \frac{2x^2 - 32}{x^3 - 4x^2} &= \lim_{x \rightarrow 4} \frac{2(x^2 - 16)}{x^2(x-4)} \\
 &= \lim_{x \rightarrow 4} \frac{2(x+4)(x-4)}{x^2(x-4)} \\
 &= \lim_{x \rightarrow 4} \frac{2(x+4)}{x^2} \\
 &= \frac{2(4+4)}{(4)^2} = \frac{2(8)}{16} \\
 &= \frac{16}{16} = 1 \quad \text{Ans.}
 \end{aligned}$$

(vii) $\lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \left(\frac{0}{0} \right) \text{ form} \quad (\text{Guj. Board 2006})$

$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} &= \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \times \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} \\
 &= \lim_{x \rightarrow 2} \frac{(\sqrt{x})^2 - (\sqrt{2})^2}{(x-2)(\sqrt{x} + \sqrt{2})} \\
 &= \lim_{x \rightarrow 2} \frac{x - 2}{(x-2)(\sqrt{x} + \sqrt{2})} \\
 &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} \quad \text{Ans.}
 \end{aligned}$$

(viii) $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ $\left(\frac{0}{0}\right)$ form (Lahore Board 2006)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \quad \text{Ans.} \end{aligned}$$

(ix) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m}$ $\left(\frac{0}{0}\right)$ form

We know that:

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}, \text{ where } n \text{ is an integer and } a > 0$$

Now,

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m} &= \lim_{x \rightarrow 0} \frac{\frac{x^n - a^n}{x - a}}{\frac{x^m - a^m}{x - a}} \\ &= \frac{na^{n-1}}{ma^{m-1}} = \frac{n}{m} a^{n-1-m+1} = \frac{n}{m} a^{n-m} \quad \text{Ans.} \end{aligned}$$

Q.3 Evaluate the following limits:

(i) $\lim_{x \rightarrow 0} \frac{\sin 7x}{x}$

(ii) $\lim_{x \rightarrow 0} \frac{\sin x^o}{x}$ (L.B 2003)

(iii) $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta}$ (L.B 2009 (s))

(iv) $\lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$

(v) $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$

(vi) $\lim_{x \rightarrow 0} \frac{x}{\tan x}$

(vii) $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$

(viii) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin^2 x}$ (L.B 2009)

(ix) $\lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta}$ (L.B 2007)

(x) $\lim_{x \rightarrow 0} \frac{\sec x - \cos x}{x}$

(xi) $\lim_{\theta \rightarrow 0} \frac{1 - \cos p\theta}{1 - \cos q\theta}$ (L.B 2004, 06) (G.B 2005, 2006)

(xii) $\lim_{\theta \rightarrow 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta}$ (L.B 2003, 2004) (G.B 2005)

Solution:

(i) $\lim_{x \rightarrow 0} \frac{\sin 7x}{x} \quad \left(\frac{0}{0} \right) \text{ form}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 7x}{x} &= \lim_{x \rightarrow 0} \frac{\sin 7x}{7x} \times 7 \\ &= 1 \times 7 = 7 \quad \text{Ans.} \end{aligned}$$

(ii) $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} \quad \left(\frac{0}{0} \right) \text{ from}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} &= \lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{180}}{\frac{x\pi}{180}} \times \frac{\pi}{180} \\ &= 1 \times \frac{\pi}{180} = \frac{\pi}{180} \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} \therefore 1^\circ &= \frac{\pi}{180} \text{ radian} \\ x^\circ &= \frac{x\pi}{180} \text{ radian} \end{aligned}$$

(iii) $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} \quad \left(\frac{0}{0} \right) \text{ form}$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} \times \frac{1 + \cos \theta}{1 + \cos \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin \theta (1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin \theta (1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{1 + \cos \theta} \\ &= \frac{0}{1 + 1} = \frac{0}{2} = 0 \quad \text{Ans.} \end{aligned}$$



(iv) $\lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$ $\left(\frac{0}{0}\right)$ form

Put $\pi - x = t \Rightarrow x = \pi - t$

As $x \rightarrow \pi$, $t \rightarrow 0$

$$\begin{aligned}\lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x} &= \lim_{t \rightarrow 0} \frac{\sin(\pi - t)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\sin t}{t} \\ &= 1 \quad \text{Ans.}\end{aligned}$$

(v) $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$ $\left(\frac{0}{0}\right)$ form (G.B 2007)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \lim_{x \rightarrow 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx} \\ &= \lim_{x \rightarrow 0} \frac{\frac{\sin ax}{ax} \times a}{\frac{\sin bx}{bx} \times b} \\ &= \frac{1 \times a}{1 \times b} = \frac{a}{b} \quad \text{Ans.}\end{aligned}$$

(vi) $\lim_{x \rightarrow 0} \frac{x}{\tan x}$ $\left(\frac{0}{0}\right)$ form (L.B 2008)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x}{\tan x} &= \lim_{x \rightarrow 0} \frac{x}{\frac{\sin x}{\cos x}} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{\frac{\sin x}{x}} \\ &= \frac{1}{1} = 1 \quad \text{Ans.}\end{aligned}$$

(vii) $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$ $\left(\frac{0}{0}\right)$ form

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \quad (\because \cos 2x = 1 - 2 \sin^2 x \Rightarrow 2 \sin^2 x = 1 - \cos 2x)$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} 2\left(\frac{\sin x}{x}\right)^2 \\
 &= 2(1)^2 = 2 \quad \text{Ans.}
 \end{aligned}$$

(viii) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin^2 x}$ $\left(\frac{0}{0}\right)$ form

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin^2 x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{1 - \cos^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{(1 + \cos x)(1 - \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\
 &= \frac{1}{1 + 1} = \frac{1}{2} \quad \text{Ans.}
 \end{aligned}$$

(ix) $\lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta}$ $\left(\frac{0}{0}\right)$ form

$$\begin{aligned}
 \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \times \sin \theta \\
 &= 1 \times 0 = 0 \quad \text{Ans.}
 \end{aligned}$$

(x) $\lim_{x \rightarrow 0} \frac{\sec x - \cos x}{x}$ $\left(\frac{0}{0}\right)$ form (G.B 2007)

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sec x - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} - \cos x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{1 - \cos^2 x}{\cos x}}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin^2 x}{\cos x}}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \tan x = 1 \times 0 = 0 \quad \text{Ans.}
 \end{aligned}$$

(xi) $\lim_{\theta \rightarrow 0} \frac{1 - \cos p\theta}{1 - \cos q\theta}$ $\left(\frac{0}{0}\right)$ form (G.B 2006)

We know that:

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$$

$$\cosp \theta = 1 - 2 \sin^2 \frac{p\theta}{2}$$

$$2\sin^2 \frac{p\theta}{2} = 1 - \cos p\theta \quad \text{and} \quad 2\sin^2 \frac{q\theta}{2} = 1 - \cos q\theta$$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos p\theta}{1 - \cos q\theta} &= \lim_{\theta \rightarrow 0} \frac{2 \sin^2 \frac{p\theta}{2}}{2 \sin^2 \frac{q\theta}{2}} \\ &= \lim_{\theta \rightarrow 0} \frac{\left(\frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \times \frac{p\theta}{2} \right)^2}{\left(\frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \times \frac{q\theta}{2} \right)^2} \\ &= \lim_{\theta \rightarrow 0} \frac{\left(\frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot \frac{p^2 \theta^2}{4}}{\left(\frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2 \cdot \frac{q^2 \theta^2}{4}} = \lim_{\theta \rightarrow 0} \frac{\left(\frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot p^2}{\left(\frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2 \cdot q^2} \\ &= \frac{(1)^2 \cdot p^2}{(1)^2 \cdot q^2} = \frac{p^2}{q^2} \quad \text{Ans.} \end{aligned}$$

(xii) $\lim_{\theta \rightarrow 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta} \quad \left(\frac{0}{0} \right) \text{ form} \quad (\text{L.B 2005})$

$$\lim_{\theta \rightarrow 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\cos \theta} - \sin \theta}{\sin^3 \theta}$$

$$\begin{aligned}
 &= \lim_{\theta \rightarrow 0} \frac{\sin\theta \left(\frac{1}{\cos\theta} - 1 \right)}{\sin^3\theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{1 - \cos\theta}{\cos\theta \sin^2\theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{1 - \cos\theta}{\cos\theta (1 - \cos^2\theta)} \\
 &= \lim_{\theta \rightarrow 0} \frac{1 - \cos\theta}{\cos\theta(1 + \cos\theta)(1 - \cos\theta)} \\
 &= \lim_{\theta \rightarrow 0} \frac{1}{\cos\theta(1 + \cos\theta)} = \frac{1}{1(1 + 1)} = \frac{1}{2} \quad \text{Ans.}
 \end{aligned}$$

Q.4 Express each limit in terms of e:

- | | |
|--|---|
| (i) $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{2n}$
(iii) $\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n}\right)^n$
(v) $\lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^n$
(vii) $\lim_{x \rightarrow 0} (1 + 2x^2)^{1/x^2}$
(ix) $\lim_{x \rightarrow \infty} \left(\frac{x}{1+x}\right)^x$ (L.B 2003,04) | (ii) $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{n/2}$
(iv) $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^n$
(vi) $\lim_{x \rightarrow 0} (1 + 3x)^{2/x}$
(viii) $\lim_{h \rightarrow 0} (1 - 2h)^{1/h}$
(x) $\lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$, $x < 0$
(xi) $\lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$, $x > 0$ (L.B 2005) |
|--|---|

Solution:

$$\begin{aligned}
 \text{(i)} \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{2n} &= \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{n}\right)^n \right]^2 \\
 &= e^2 \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{n/2} &= \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{n}\right)^n \right]^{1/2} \\
 &= e^{1/2} \quad \text{Ans.}
 \end{aligned}$$

$$\text{(iii)} \quad \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \rightarrow +\infty} \left[\left(1 + \left(\frac{1}{-n}\right)\right)^{-n} \right]^{-1}$$

$$= e^{-1} \quad \text{Ans.}$$

$$\text{(iv)} \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^n = \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{3n}\right)^{3n} \right]^{1/3}$$

$$= e^{1/3} \quad \text{Ans.}$$

$$\text{(v)} \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^n = \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{n/4}\right)^{n/4} \right]^4$$

$$= e^4 \quad \text{Ans.}$$

$$\text{(vi)} \quad \lim_{n \rightarrow 0} (1 + 3x)^{2/x} = \lim_{n \rightarrow 0} [(1 + 3x)^{1/(3x)}]^{2 \times 3}$$

$$= e^6 \quad \text{Ans.}$$

$$\text{(vii)} \quad \lim_{x \rightarrow 0} (1 + 2x^2)^{1/x^2} = \lim_{x \rightarrow 0} [(1 + 2x^2)^{1/(2x^2)}]^2$$

$$= e^2 \quad \text{Ans.}$$

$$\text{(viii)} \quad \lim_{h \rightarrow 0} (1 - 2h)^{1/h} = \lim_{h \rightarrow 0} [(1 + (-2h)^{-1/2h})]^{+2}$$

$$= e^{-2} \quad \text{Ans.}$$

$$\text{(ix)} \quad \lim_{x \rightarrow \infty} \left(\frac{x}{1+x}\right)^x = \lim_{x \rightarrow \infty} \left(\frac{1+x}{x}\right)^{-x} \quad (\text{G.B 2006}) \quad (\text{L.B 2007})$$

$$= \lim_{x \rightarrow \infty} \left(\frac{1}{x} + \frac{1}{x}\right)^{-x}$$

$$= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{x}\right)^x \right]^{-1} = e^{-1} \quad \text{Ans.}$$

$$\text{(x)} \quad \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}, \quad x < 0 \quad (\text{G.B 2005})$$

Put, $x = -t$, where $t > 0$

As, $x \rightarrow 0$, $t \rightarrow 0$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} &= \lim_{t \rightarrow 0} \frac{e^{1/-t} - 1}{e^{1/-t} + 1} \\ &= \frac{e^{-1/0} - 1}{e^{-1/0} + 1} = \frac{e^{-\infty} - 1}{e^{-\infty} + 1} \\ &= \frac{0 - 1}{0 + 1} = \frac{-1}{1} = -1 \quad \text{Ans.} \end{aligned}$$

$$(xi) \quad \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}, \quad x > 0 \quad \left(\frac{\infty}{\infty} \right)$$

$$\lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{x \rightarrow 0} \frac{e^{1/x} (1 - \frac{1}{e^{1/x}})}{e^{1/x} (1 + \frac{1}{e^{1/x}})}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}}$$

$$= \frac{1 - \frac{1}{e^{\infty}}}{1 + \frac{1}{e^{\infty}}}$$

$$= \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1 \quad \text{Ans.}$$

Continuous Function

A function f is said to be continuous at a number “ c ” if and only if the following three conditions are satisfied.

- (i) $f(c)$ is defined.
- (ii) $\lim_{x \rightarrow c} f(x)$ exists.
- (iii) $\lim_{x \rightarrow c} f(x) = f(c)$

EXERCISE 1.4

Q.1 Determine the left hand limit and right hand limit and then find limits of the following functions at $x = c$.

- (i) $f(x) = 2x^2 + x - 5, \quad c = 1$
- (ii) $f(x) = \frac{x^2 - 9}{x - 3}, \quad c = -3$
- (iii) $f(x) = |x - 5|, \quad c = 5$

Solution:

(i) $f(x) = 2x^2 + x - 5, \quad c = 1$

Left hand limit