

$$(xi) \quad \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}, \quad x > 0 \quad \left( \frac{\infty}{\infty} \right)$$

$$\lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{x \rightarrow 0} \frac{e^{1/x} (1 - \frac{1}{e^{1/x}})}{e^{1/x} (1 + \frac{1}{e^{1/x}})}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}}$$

$$= \frac{1 - \frac{1}{e^{\infty}}}{1 + \frac{1}{e^{\infty}}}$$

$$= \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1 \quad \text{Ans.}$$

### Continuous Function

A function  $f$  is said to be continuous at a number “ $c$ ” if and only if the following three conditions are satisfied.

- (i)  $f(c)$  is defined.
- (ii)  $\lim_{x \rightarrow c} f(x)$  exists.
- (iii)  $\lim_{x \rightarrow c} f(x) = f(c)$

### EXERCISE 1.4

**Q.1 Determine the left hand limit and right hand limit and then find limits of the following functions at  $x = c$ .**

- (i)  $f(x) = 2x^2 + x - 5, \quad c = 1$
- (ii)  $f(x) = \frac{x^2 - 9}{x - 3}, \quad c = -3$
- (iii)  $f(x) = |x - 5|, \quad c = 5$

**Solution:**

(i)  $f(x) = 2x^2 + x - 5, \quad c = 1$

Left hand limit

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (2x^2 + x - 5) \\ &= 2(1)^2 + 1 - 5 \\ &= 2 - 4 = -2 \quad \text{Ans.}\end{aligned}$$

Right hand limit

$$\begin{aligned}\lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (2x^2 + x - 5) \\ &= 2(1)^2 + 1 - 5 \\ &= 2 + 1 - 5 \\ &= -2 \quad \text{Ans.}\end{aligned}$$

(ii)  $f(x) = \frac{x^2 - 9}{x - 3}$  ,  $c = -3$

Left hand limit

$$\begin{aligned}\lim_{x \rightarrow -3^-} f(x) &= \lim_{x \rightarrow -3^-} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow -3^-} \frac{(x + 3)(x - 3)}{x - 3} \\ &= \lim_{x \rightarrow -3^-} (x + 3) \\ &= -3 + 3 = 0 \quad \text{Ans.}\end{aligned}$$

Right hand limit

$$\begin{aligned}\lim_{x \rightarrow -3^+} f(x) &= \lim_{x \rightarrow -3^+} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow -3^+} \frac{(x + 3)(x - 3)}{x - 3} \\ &= \lim_{x \rightarrow -3^+} (x + 3) \\ &= -3 + 3 = 0 \quad \text{Ans.}\end{aligned}$$

(iii)  $f(x) = |x - 5|$  ,  $c = 5$

Left hand limit

$$\begin{aligned}\lim_{x \rightarrow 5^-} f(x) &= \lim_{x \rightarrow 5^-} |x - 5| \\ &= \lim_{x \rightarrow 5^-} -(x - 5) \\ &= -(5 - 5) = 0 \quad \text{Ans.}\end{aligned}$$

Right hand limit

$$\begin{aligned}\lim_{x \rightarrow 5^+} f(x) &= \lim_{x \rightarrow 5^+} |x - 5| \\ &= \lim_{x \rightarrow 5^+} (x - 5) \\ &= 5 - 5 \\ &= 0 \quad \text{Ans.}\end{aligned}$$

**Q.2 Discuss the continuity of  $f(x)$  at  $x = c$ :**

$$\begin{array}{lll} \text{(i)} \quad f(x) = , \quad c = 2 & & (\text{G.B 2007, L.B 2008}) \\ \text{(ii)} \quad f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}, \quad c = 1 & & (\text{L.B 2009, L.B 2006}) \\ & & (\text{L.B 2009, G.B 2007}) \end{array}$$

**Solution:**

$$\begin{array}{ll} \text{(i)} \quad f(x) = \begin{cases} 2x + 5 & \text{if } x \leq 2 \\ 4x + 1 & \text{if } x > 2 \end{cases}, \quad c = 2 \\ f(2) = 2(2) + 5 \\ = 4 + 5 \\ = 9 \end{array}$$

Left hand limit

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (2x + 5) \\ &= 2(2) + 5 \\ &= 4 + 5 = 9\end{aligned}$$



Right hand limit

$$\begin{aligned}\lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (4x + 1) \\ &= 4(2) + 1 \\ &= 8 + 1 \\ &= 9\end{aligned}$$

$\therefore$  Left hand limit = Right hand limit

So  $\lim_{x \rightarrow 2} f(x)$  exists

$$\therefore f(2) = \lim_{x \rightarrow 2} f(x) = 9$$

So the function is continuous at  $x = 2$ .

$$(ii) \quad f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}, \quad c = 1 \quad (L.B 2006, 2007)$$

$$f(1) = 4$$

Left hand limit

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (3x - 1) \\ &= 3(1) - 1 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

Right hand limit

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (2x) \\ &= 2(1) = 2 \end{aligned}$$

$\therefore$  Left hand limit = Right hand limit

So  $\lim_{x \rightarrow 1} f(x)$  exists

$$\therefore f(1) \neq \lim_{x \rightarrow 1} f(x)$$

So the function is discontinuous at  $x = 1$ .

$$Q.3 \quad \text{If } f(x) = \begin{cases} 3x & \text{if } x \leq -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases} \quad (L.B 2011)$$

Discuss continuity at  $x = 2$  and  $x = -2$ .

**Solution:**

$$\text{At } x = 2$$

$$f(2) = 3$$

Left hand limit

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (x^2 - 1) \\ &= 2^2 - 1 = 4 - 1 = 3 \end{aligned}$$

Right hand limit

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} 3 \\ &= 3 \end{aligned}$$

$\therefore$  Left hand limit = Right hand limit

So,  $\lim_{x \rightarrow 2} f(x)$  exists

$$\therefore f(2) = \lim_{x \rightarrow 2} f(x) = 3$$

So the function is continuous at  $x = 2$ .

At  $x = -2$

$$f(-2) = 3(-2) = -6$$

Left hand limit.

$$\begin{aligned}\lim_{x \rightarrow -2^-} f(x) &= \lim_{x \rightarrow -2^-} (3x) \\ &= 3(-2) = -6\end{aligned}$$

Right hand limit.

$$\begin{aligned}\lim_{x \rightarrow -2^+} f(x) &= \lim_{x \rightarrow -2^+} (x^2 - 1) \\ &= (-2)^2 - 1 \\ &= 4 - 1 \\ &= 3\end{aligned}$$

$\therefore$  Left hand limit  $\neq$  Right hand limit

So,  $\lim_{x \rightarrow -2} f(x)$  does not exist.

$$\therefore f(-2) \neq \lim_{x \rightarrow -2} f(x)$$

So the function is discontinuous at  $x = -2$ .

**Q.4** If  $f(x) = \begin{cases} x + 2, & x \leq -1 \\ c + 2, & x > -1 \end{cases}$  find 'c' so that  $\lim_{x \rightarrow -1} f(x)$  exists. (L.B 2009 Supply)  
(G.B 2008)

**Solution:**

Left hand limit

$$\begin{aligned}\lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} (x + 2) \\ &= -1 + 2 = 1\end{aligned}$$

Right hand limit

$$\begin{aligned}\lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} (c + 2) \\ &= c + 2\end{aligned}$$

Since  $\lim_{x \rightarrow -1} f(x)$  exists.

$\therefore$  Left hand limit = Right hand limit

$$1 = c + 2$$

$$c = 1 - 2$$

$c = -1$	Ans.
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**Q.5 Find the values m and n, So that given function f is continuous at x = 3:**

$$(i) \quad f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 9 & \text{if } x > 3 \end{cases} \quad (ii) \quad f(x) = \begin{cases} mx & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$$

**Solution:**

$$(i) \quad f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 9 & \text{if } x > 3 \end{cases} \quad (L.B 2004, 2005) \quad (G.B 2006, 2009)$$

$$f(3) = n$$

Left hand limit

$$\begin{aligned} \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} (mx) \\ &= 3m \end{aligned}$$

Right hand limit

$$\begin{aligned} \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} (-2x + 9) \\ &= -2(3) + 9 \\ &= -6 + 9 \\ &= 3 \end{aligned}$$

Since f(x) is continuous at x = 3

$$\therefore \text{Left hand limit} = \text{Right hand limit} = f(3)$$

$$3m = 3 = n$$

$$3m = 3, 3 = n$$

$$m = \frac{3}{3} \quad n = 3$$

$$m = 1$$

$$\boxed{\therefore m = 1, n = 3} \quad \text{Ans.}$$

$$(ii) \quad f(x) = \begin{cases} mx & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases} \quad (L.B 2007)$$

$$f(3) = (3)^2 = 9$$

Left hand limit

$$\begin{aligned} \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} (mx) \\ &= 3m \end{aligned}$$

Right hand limit

$$\begin{aligned} \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} (x^2) \\ &= 3^2 = 9 \end{aligned}$$

Since f(x) is continuous at x = 3

$$\therefore \text{Left hand limit} = \text{Right hand limit} = f(3)$$

$$3m = 9 = 9$$

$$3m = 9$$

$$m = \frac{9}{3} = 3 \quad \text{Ans.}$$

Q.6: If  $f(x) = \begin{cases} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

(G.B 2004)

(L.B 2009 (s) 2004)

(G.B 2006)

(L.B 2008)

(G.B 2008)

*Find value of k so that f is continuous at x = 2.***Solution:**

$$f(2) = k$$

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \left( \frac{0}{0} \right) \text{ form} \\ &= \lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} \\ &= \lim_{x \rightarrow 2} \frac{(\sqrt{2x+5})^2 - (\sqrt{x+7})^2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} \\ &= \lim_{x \rightarrow 2} \frac{(2x+5) - (x+7)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} \\ &= \lim_{x \rightarrow 2} \frac{\frac{x-2}{x-2}}{(\sqrt{2x+5} + \sqrt{x+7})} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} \\ &= \frac{1}{\sqrt{2(2)+5} + \sqrt{2+7}} \\ &= \frac{1}{\sqrt{4+5} + \sqrt{9}} \\ &= \frac{1}{3+3} = \frac{1}{6} \end{aligned}$$

Since  $f(x)$  is continuous at  $x = 2$ 

$$\therefore f(2) = \lim_{x \rightarrow 2} f(x)$$

$k = \frac{1}{6}$
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Ans.