EXERCISE 2.10

Q.1: Find two positive integers whose sum is 30 and their product will be maximum.

Solution:

Let 1^{st} positive integer $= x$

2^{nd} positive integer $= 30 - x$

According to the condition

Product $= P = x(30 - x)$

$P = 30x - x^2$

$$\frac{dP}{dx} = 30 - 2x$$

$$\frac{d^2P}{dx^2} = -2$$

For stationary points

Put

$$\frac{dP}{dx} = 0$$

$$30 - 2x = 0$$

$$30 = 2x$$

$$x = \frac{30}{2} = 15$$

Put $x = 15$ in $\frac{d^2P}{dx^2}$, we get

$$\frac{d^2P}{dx^2} = -2 < 0$$

$\therefore$ Product is maximum at $x = 15$

1^{st} positive integer $= x = 15$

2^{nd} positive integer $= 30 - x$

$= 30 - 15$

$= 15$ Ans.

Q.2: Divide 20 into two parts so that the sum of their squares will be minimum.

Solution:

Let

1^{st} part $= x$

2^{nd} part $= 20 - x$

Sum of square $= S = x^2 + (20 - x)^2$

$$S = x^2 + 400 + x^2 - 40x$$

$$S = 2x^2 - 40x + 400$$

$$\frac{dS}{dx} = 4x - 40$$

$$\frac{d^2S}{dx^2} = 4$$

For stationary points

Put

$$\frac{dS}{dx} = 0$$

$$4x - 40 = 0$$

$$4x = 40$$

$$x = \frac{40}{4} = 10$$

Put $x = 10$ in $\frac{d^2S}{dx^2}$, we get

$$\frac{d^2S}{dx^2} = 4 > 0$$

$\therefore$ Sum of squares is minimum at $x = 10$

1^{st} part $= x = 10$

2^{nd} part $= 20 - x$

$= 20 - 10$

$= 10$ Ans.

Q.3: Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum. ***(L.B 2009)***

Solution:

Let

1^{st} positive integer $= x$

2^{nd} positive integer $= 12 - x$

According to the condition

Product $= P = x(12 - x)^2$

$$P = x(144 + x^2 - 24x)$$

$$P = 144x + x^3 - 24x^2$$

$$\frac{dP}{dx} = 144x + 3x^2 - 48x$$

$$\frac{d^2P}{dx^2} = 6x - 48$$

For stationary points

Put $\frac{dP}{dx} = 0$

$$144 + 3x^2 - 48x = 0$$
$$3(x^2 - 16x + 48) = 0$$
$$x^2 - 16x + 48 = 0$$
$$x^2 - 12x - 4x + 48 = 0$$
$$x(x - 12) - 4(x - 12) = 0$$
$$(x - 12)(x - 4) = 0$$

Either

$x - 12 = 0$ or $x - 4 = 0$

$x = 12$ $\quad$ $x = 4$

Put

$x = 12$ in $\frac{d^2P}{dx^2}$, we gt

$$\frac{d^2P}{dx^2} = 6(12) - 48$$
$$= 72 - 48$$
$$= 24 > 0$$

This is not possible because product will be maximum.

Put $x = 4$ in $\frac{d^2P}{dx^2}$, we get

$$\frac{d^2P}{dx^2} = 6(4) - 48$$
$$= 24 - 48$$
$$= -24 < 0$$

$\therefore$ Product is maximum at $x = 4$

1st positive integer $= x = 4$

2nd positive integer $= 12 - x$

$= 12 - 4 = 8$ Ans.

Q.4: The perimeter of a triangle is 16 centimeters. If one side of length 6cm, what are length of the other sides for maximum area of the triangle?

Solution:

Length of 1^{st} side of triangle = 6 cm

Let

Length of 2^{nd} side of triangle = x cm

Length of 3^{rd} side of triangle $= 16 - (6 + x)$

$= 16 - 6 - x$

$= (10 - x)$ cm

$$S = \frac{6 + x + 10 - x}{2}$$

$$S = \frac{16}{2} = 8 \text{ cm}$$

$$\text{Area} = A = \sqrt{S(S-a)(S-b)(S-c)}$$

$$A = \sqrt{8(8-6)(8-x)(8-10+x)}$$

$$A = \sqrt{8(2)(8-x)(x-2)}$$

$$A = \sqrt{16(8x - 16 - x^2 + 2x)}$$

$$A = 4\sqrt{-x^2 + 10 - 16}$$

$$\frac{dA}{dx} = 4.\frac{1}{2}.(-x^2 + 10x - 16)^{\frac{-1}{2}}(-2x + 10)$$

$$\frac{dA}{dx} = \frac{2(-2x + 10)}{\sqrt{-x^2 + 10x - 6}}$$

For stationary points

Put

$$\frac{dA}{dx} = 0$$

$$\frac{2(-2x + 10)}{\sqrt{-x^2 + 10x - 16}} = 0$$

$$4(-x + 5) = 0$$

$$-x + 5 = 0$$

$$-x = -5$$

$$\boxed{x = 5}$$

Before $x = 5$, $\frac{dA}{dx} > 0$

After $x = 5$, $\frac{dA}{dx} < 0$

$\therefore$ Area is maximum at $x = 5$

1^{st} side of triangle = 6 cm

2nd side of triangle $= x = 5$ cm

3rd side of triangle $= 10 - x = 5$ cm Ans.

Q.5: Find the dimensions of a rectangle of largest area having perimeter 120 centimetres.

Solution:

Let

Length of rectangle $= x$ cm

Width of rectangle $= y$ cm

Perimeter $= 120$ cm

$$2(x + y) = 120$$

$$x + y = \frac{120}{2}$$

$$x + y = 60$$

$$y = 60 - x \quad \text{..... (1)}$$

$$\text{Area} = A = xy$$

$$A = x(60 - x)$$

$$A = 60x - x^2$$

$$\frac{dA}{dx} = 60 - 2x$$

$$\frac{d^2A}{dx^2} = -2$$

For Stationary points

Put

$$\frac{dA}{dx} = 0$$

$$60 - 2x = 0$$

$$-2x = -60$$

$$x = \frac{60}{2} = 30$$

Put $x = 30$ in $\frac{d^2A}{dx^2}$, we get

$$\frac{d^2A}{dx^2} = -2 < 0$$

$\therefore$ Area is maximum at $x = 30$

Put

$x = 30$ cm in eq. (1)

$$y = 60 - 30 = 30 \text{ cm}$$

Length of rectangle $= x = 30$ cm

Width of rectangle $= y = 30$ cm

Dimensions of rectangle are 30 cm, 30 cm

Q.6: Find the lengths of the sides of a variable rectangle having area 36 cm^2 when its perimeter is minimum.

Solution:

Let

Length of rectangle $= x$ cm

Width of rectangle $= y$ cm

Area $= A = 36 \text{ cm}^2$

$$xy = 36$$

$$y = \frac{36}{x} \quad \text{..............} (1)$$

Perimeter $= P = 2(x + y)$

$$P = 2\left(x + \frac{36}{x}\right)$$

$$\frac{dP}{dx} = 2\left(1 - \frac{36}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 2\left(0 + \frac{72}{x^3}\right)$$

$$= \frac{144}{x^3}$$

For stationary points

Put $\frac{dP}{dx} = 0$

$$2\left(1 - \frac{36}{x^2}\right) = 0$$

$$1 - \frac{36}{x^2} = 0$$

$$\frac{x^2 - 36}{x^2} = 0$$

$$x^2 - 36 = 0$$

$$x^2 = 36$$

$$x = 6$$

Put $x = 6$ in $\frac{d^2P}{dx^2}$, we get

$$\frac{d^2P}{dx^2} = \frac{144}{(6)^3} = \frac{144}{216} > 0$$

$\therefore$ Perimeter is minimum at $x = 6$ cm

Put $x = 6$ cm in eq. (1)

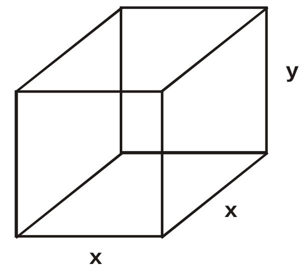
$$y = \frac{36}{6} = 6 \text{ cm}$$

Length of rectangle $= x = 6$ cm

Width of rectangle $= y = 6$ cm Ans.

Q.7: A box with a square base and open top is to have a volume of 4 cubic dm. Find the dimensions of the box which will require the least material.

Solution:



Let Length of box $= x$

Width of box $= x$

Height of box $= y$

Volume $=$ 4 cubic dm

$$x \times x \times y = 4$$

$$x^2 y = 4$$

$$y = \frac{4}{x^2} \quad \ldots\ldots\ (1)$$

Surface area of box $= S = x^2 + xy + xy + xy + xy$

$$= x^2 + 4xy$$

$$= x^2 + 4x\left(\frac{4}{x^2}\right)$$

$$S = x^2 + \frac{16}{x}$$

$$\frac{dS}{dx} = 2x - \frac{16}{x^2}$$

$$\frac{d^2S}{dx^2} = 2 + \frac{32}{x^3}$$

For stationery points

Put $\frac{dS}{dx} = 0$

$$2x - \frac{16}{x^2} = 0$$

$$\frac{2x^3 - 16}{x^2} = 0$$

$$2x^3 - 16 = 0$$

$$2x^3 = 16$$

$$x^3 = 8$$
$$x = 2$$

Put $x = 2$ in $\frac{d^2S}{dx^2}$, we get

$$\frac{d^2S}{dx^2} = 2 + \frac{32}{(2)^3}$$
$$= 2 + \frac{32}{8}$$
$$= 2 + 4$$
$$= 12 > 0$$

$\therefore$ Surface area is minimum at $x = 2$

Put $x = 2$ in

$$y = \frac{4}{x^2} = \frac{4}{(2)^2} = \frac{4}{4} = 1$$

Length of box $= x = 2$dm

Width of box $= x = 2$dm

Height of box $= y = 1$dm Ans.

Q.8: Find the dimensions of a rectangular garden having perimeter 80 metres if its area is to be maximum.

Solution:

Let Length of rectangular garden $= x$ m

Width of rectangular garden $= y$ m

Perimeter $= 80$ m

$$2(x + y) = \frac{80}{2}$$

$$y = 40 - x \quad .... (1)$$

Area $= A = xy$

$$A = x(40 - x)$$
$$A = 40x - x^2$$
$$\frac{dA}{dx} = 40 - 2x$$
$$\frac{d^2A}{dx^2} = -2$$

For stationary points

Put $\frac{dA}{dx} = 0$

$$40 - 2x = 0$$
$$-2x = -40$$
$$x = \frac{-40}{-2} = 20$$

Put $x = 20$ in $\frac{d^2A}{dx^2}$, we get

$$\frac{d^2A}{dx^2} = -2 < 0$$

$\therefore$ Area is maximum at $x = 20$

Put $x = 20$ in eq. (1)

$$y = 40 - x$$
$$y = 40 - 20 = 20$$

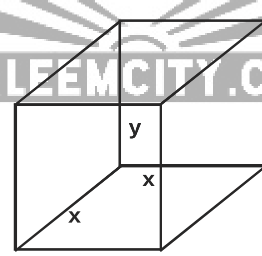
$\therefore$ Length of rectangular garden $= x = 20$ m

Width of rectangular garden $= y = 20$ m

So the dimensions of the rectangular garden are 20 m, 20 m.

Q.9: An open tank of square base of side x and vertical sides is to be constructed to contain of given quantity of water. Find the depth in terms of x if the expense of lining the inside of the tank with will be least.

Solution:



Let

Length of tank $= x$

Width of tank $= x$

Height of tank $= y$

Let 'q' be the quantity of water in the tank.

Volume $= V = q$

$x \times x \times y = q$

$$y = \frac{q}{x^2} \quad \text{............ (1)}$$

Total surface area $= S = x^2 + xy + xy + xy + xy$

$$S = x^2 + 4xy$$

$$S = x^2 + 4x\left(\frac{q}{x^2}\right)$$

$$S = x^2 + \frac{4q}{x^2}$$

$$\frac{dS}{dx} = 2x - \frac{4q}{x^2}$$

$$\frac{d^2S}{dx^2} = 2x + \frac{8q}{x^3}$$

For stationary points

Put $\frac{dS}{dx} = 0$

$$2x - \frac{4q}{x^2} = 0$$

$$\frac{2x^3 - 4q}{x^2} = 0$$

$$2x^3 - 4q = 0$$

$$2x^3 = 4q$$

$$x^3 = \frac{4q}{2}$$

$$x^3 = 2q$$

$$x = (2q)^{\frac{1}{3}}$$

Put $x = (2q)^{\frac{1}{3}}$ in

$$\frac{d^2S}{dx^2} = 2 + \frac{8q}{[(2q)^{\frac{1}{3}}]^3}$$

$$= 2 + \frac{8q}{2q}$$

$$= 2 + 4$$

$$= 6 > 0$$

$\therefore$ Surface area is minimum at $x = (2q)^{\frac{1}{3}}$

Now

$$x^3 = 2q$$

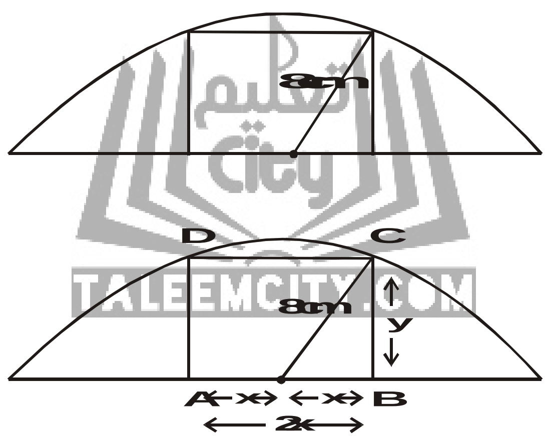
$$q = \frac{x^3}{2}$$

Put $q = \dfrac{x^3}{2}$ in eq. (1)

$$y = \frac{\frac{x^3}{2}}{x^2}$$

$$\boxed{\text{Depth} = y = \frac{x}{2}}$$ Ans

Q.10: Find the dimensions of the rectangle of maximum area which fits, inside the semi-circle of radius 8 cm as shown in the figure.



Solution:

Let

Length of rectangle = 2x cm

Width of rectangle = y cm

From right angle Δ EBC

$$(8)^2 = x^2 + y^2$$

$$y^2 = 64 - x^2$$

$$y = \sqrt{64 - x^2} \quad \text{... (1)}$$

$$\text{Area} = A = (2x)(y)$$

$$A = 2x\sqrt{64 - x^2}$$

Diff. w.r.t. 'x'

$$\frac{dA}{dx} = 2\,[x \,.\, \frac{1}{2}\,(64 - x^2)^{\frac{-1}{2}} - 2x + \sqrt{64 - x^2}\,]$$

$$\frac{dA}{dx} = 2\left[\frac{-x^2}{\sqrt{64 - x^2}} + \sqrt{64 - x^2}\right]$$

$$\frac{dA}{dx} = 2\left[\frac{-x^2 + 64 - x^2}{\sqrt{64 - x^2}}\right]$$

$$\frac{dA}{dx} = 2\left(\frac{64 - 2x^2}{\sqrt{64 - x^2}}\right)$$

For stationary points

Put $\frac{dA}{dx} = 0$

$$2\left(\frac{64 - 2x^2}{\sqrt{64 - x^2}}\right) = 0$$

$$64 - 2x^2 = 0$$

$$64 = 2x^2$$

$$x^2 = \frac{64}{2}$$

$$x^2 = 32$$

$$x = \sqrt{32}$$

$$x = 4\sqrt{2}$$

Before $x = 4\sqrt{2}$, $\frac{dA}{dx} > 0$

After $x = 4\sqrt{2}$, $\frac{dA}{dx} < 0$

$\therefore$ Area is maximum at $x = 4\sqrt{2}$

Put $x = 4\sqrt{2}$ in eq. (1)

$$y = \sqrt{64 - (4\sqrt{2})^2}$$

$$= \sqrt{64 - 32}$$

$$= \sqrt{32} = 4\sqrt{2}$$

Length of rectangle $= x = 4\sqrt{2}$ cm

Width of rectangle $= y = 4\sqrt{2}$ cm

So the dimensions of rectangle are $4\sqrt{2}$ cm, $4\sqrt{2}$ cm

Q.11: Find the point on the curve $y = x^2 - 1$ that is closest to the point (3, – 1).

Solution:

$$y = x^2 - 1$$

Let P (x, y) be the point on the curve and A (3, – 1) be the given point.

Now Distance $= l = |PA|$

$$l = \sqrt{(x-3)^2 + (y+1)^2}$$

$$l = \sqrt{x^2 + 9 - 6x + (x^2 - 1 + 1)^2}$$

$$l = \sqrt{x^2 + 9 - 6x + x^4}$$

$$l = \sqrt{x^4 + x^2 - 6x + 9}$$

$$\frac{d\ell}{dx} = \frac{1}{2}(x^4 + x^2 - 6x + 9)^{\frac{-1}{2}}(4x^3 + 2x - 6)$$

$$= \frac{2(2x^3 + x - 3)}{2\sqrt{x^4 + x^2 - 6x + 9}}$$

$$\frac{d\ell}{dx} = \frac{2x^3 + x - 3}{\sqrt{x^4 + x^2 - 6x + 9}}$$

For stationary points

Put $\frac{d\ell}{dx} = 0$

$$\frac{2x^3 + x - 3}{\sqrt{x^4 + x^2 - 6x + 9}} = 0$$

$$2x^3 + x - 3 = 0$$

By using synthetic division

	2	0	1	– 3
1		2	2	3
	2	2	3	0

$x = 1$ and depressed equation is $2x^2 + 2x + 3 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 2, \quad b = 2, \quad c = 3$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{-2 \pm \sqrt{4 - 24}}{4}$$

$$x = \frac{-2 \pm \sqrt{-20}}{4}$$

Neglecting because it has imaginary roots.

Before $x = 1$, $\frac{d\ell}{dx} < 0$

After $x = 1$, $\frac{d\ell}{dx} > 0$

$\therefore$ Distance is minimum at $x = 1$

Put $x = 1$ in

$y = x^2 - 1$

$y = (1)^2 - 1 = 1 - 1 = 0$

$\therefore$ Point is $P(x, y) = P(1, 0)$ Ans.

Q.12: Find the point on the curve $y = x^2 + 1$ that is closest to the point (18, 1).

Solution:

$y = x^2 + 1$

Let P (x, y) be the point on the curve and A (18, 1) be the given point.

Now

Distance $= \ell = |PA|$

$$l = \sqrt{(x-18)^2 + (y-1)^2}$$

$$l = \sqrt{x^2 + 324 - 36x + (x^2 + 1 - 1)^2}$$

$$l = \sqrt{x^2 + 324 - 36x + x^4}$$

$$l = \sqrt{x^4 + x^2 - 36x + 324}$$

$$\frac{d\ell}{dx} = \frac{1}{2}\ (x^4 + x^2 - 36x + 324)^{\frac{-1}{2}}\ .\ (4x^3 + 2x - 36)$$

$$\frac{d\ell}{dx} = \frac{2\,(2x^3 + x - 18)}{2\sqrt{x^4 + x^2 - 36x + 324}}$$

$$\frac{d\ell}{dx} = \frac{2x^3 + x - 18}{\sqrt{x^4 + x^2 - 36x + 324}}$$

For stationary points

Put

$$\frac{d\ell}{dx} = 0$$

$$\frac{2x^3 + x - 18}{\sqrt{x^4 + x^2 - 36x + 324}} = 0$$

$$2x^3 + x - 18 = 0$$

By using synthetic division

$$\begin{array}{c|cccc|c} & 2 & 0 & 1 & & -18 \\ 2 & & 4 & 8 & & 8 \\ \hline & 2 & 4 & 9 & & 0 \end{array}$$

$x = 2$ and depressed equation is $2x^2 + 4x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 2, \quad b = 4, \quad c = 9$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(9)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{16 - 72}}{4}$$

$$x = \frac{-4 \pm \sqrt{-56}}{4}$$

Neglecting because it has imaginary roots.

Before $x = 2$, $\frac{d\ell}{dx} < 0$

After $x = 2$, $\frac{d\ell}{dx} > 0$

$\therefore$ Distance is minimum at $x = 2$

Put $x = 2$ in

$y = x^2 + 1$

$y = (2)^2 + 1 = 4 + 1 = 5$

$\therefore$ Point is $P(x, y) = P(2, 5)$ Ans.