

EXERCISE 2 . 4

Q.1 Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

$$(i) \quad y = \sqrt{\frac{1-x}{1+x}}$$

$$(ii) \quad y = \sqrt{x + \sqrt{x}}$$

$$(iii) \quad y = \sqrt{\frac{a+x}{a-x}}$$

$$(iv) \quad y = (3x^2 - 2x + 7)^6$$

$$(v) \quad y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}$$

Solution:

$$(i) \quad y = \sqrt{\frac{1-x}{1+x}}$$

$$\text{Let } u = \frac{1-x}{1+x} \quad \text{So } y = \sqrt{u} = u^{1/2}$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{1-x}{1+x} \right)$$

$$\frac{du}{dx} = \frac{(1+x)\frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{(1+x)(0-1) - (1-x)(0+1)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-2}{(1+x)^2}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du} (u^{1/2})$$

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{1/2}$$

$$\frac{dy}{du} = \frac{(1+x)^{1/2}}{2(1-x)^{1/2}}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{(1+x)^{1/2}}{2(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}(1+x)^{2-\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}(1+x)^{3/2}}$$

Ans.

(ii) $y = \sqrt{x + \sqrt{x}}$ (G.B 2007)

Let

$$u = x + \sqrt{x} \quad \text{So } y = \sqrt{u} = u^{1/2}$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{1/2})$$

$$\frac{du}{dx} = 1 + \frac{1}{2}x^{-1/2}$$

$$\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{2\sqrt{x} + 1}{2\sqrt{x}}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'

$$\frac{dy}{du} = \frac{d}{du}(u^{1/2})$$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} \cdot 1$$

$$\frac{dy}{du} = \frac{1}{2}(x + \sqrt{x})^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{x + \sqrt{x}}}$$

By using chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ \frac{dy}{dx} &= \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{2\sqrt{x+1}}{2\sqrt{x}} \\ \frac{dy}{dx} &= \frac{2\sqrt{x+1}}{4\sqrt{x}\sqrt{x+\sqrt{x}}} \quad \text{Ans.}\end{aligned}$$

(iii) $y = x \sqrt{\frac{a+x}{a-x}}$

Put,

$$u = \frac{a+x}{a-x} \quad \text{So} \quad y = x\sqrt{u} = xu^{1/2}$$

Now,

$$u = xu^{1/2}$$

Diff. w.r.t. 'x'.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(xu^{1/2}) \\ \frac{dy}{dx} &= x \frac{d}{dx}(u^{1/2}) + u^{1/2} \cdot \frac{d}{dx}(x) \\ \frac{dy}{dx} &= \frac{x}{2} u^{-1/2} \frac{du}{dx} + u^{1/2} \quad \dots\dots\dots (1) \\ u &= \frac{a+x}{a-x}\end{aligned}$$

Diff. w.r.t. 'x'.

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}\left(\frac{a+x}{a-x}\right) \\ \frac{du}{dx} &= \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2} \\ \frac{du}{dx} &= \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2} \\ \frac{du}{dx} &= \frac{a-x+a+x}{(a-x)^2} \\ \frac{du}{dx} &= \frac{2a}{(a-x)^2}\end{aligned}$$

\therefore From equation (1).

$$\frac{dy}{dx} = \frac{x}{2} \left(\frac{a+x}{a-x} \right)^{-1/2} \cdot \frac{2a}{(a-x)^2} + \left(\frac{a+x}{a-x} \right)^{1/2}$$

$$\frac{dy}{dx} = ax \frac{(a+x)^{-1/2}}{(a-x)^{-1/2}} \cdot \frac{1}{(a-x)^2} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x} (a-x)^{-1/2+2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x} (a-x)^{3/2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax + (a+x)(a-x)}{\sqrt{a+x} (a-x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{ax + a^2 - x^2}{\sqrt{a+x} (a-x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{a^2 - x^2 + ax}{\sqrt{a+x} (a-x)^{3/2}}$$

Ans.

(iv) $y = (3x^2 - 2x + 7)^6$ (L.B 2009 (s))

Let $u = 3x^2 - 2x + 7$ So $y = u^6$

Diff. w.r.t. 'x'.

$$\frac{du}{dx} = 3 \frac{d}{dx}(x^2) - 2 \frac{d}{dx}(x) + \frac{d}{dx}(7)$$

$$= 3(2x) - 2(1) + 0$$

$$\frac{du}{dx} = 6x - 2$$

$$y = u^6$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du}(u^6)$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{dy}{du} = 6(3x^2 - 2x + 7)^5$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 6(3x^2 - 2x + 7)^5 \cdot (6x - 2)$$

$$\boxed{\frac{dy}{dx} = 12(3x^2 - 2x + 7)^5 (3x - 1)} \quad \text{Ans.}$$

(v) $y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$ (G.B 2004)

Let $u = \frac{a^2 + x^2}{a^2 - x^2}$ So $y = \sqrt{u} = u^{1/2}$

Diff. w.r.t. 'x'.

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{a^2 + x^2}{a^2 - x^2} \right)$$

$$\frac{du}{dx} = \frac{(a^2 - x^2) \frac{d}{dx}(a^2 + x^2) - (a^2 + x^2) \frac{d}{dx}(a^2 - x^2)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{(a^2 - x^2)2x - (a^2 + x^2) \cdot (-2x)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{2x(a^2 - x^2 + a^2 + x^2)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{2x(2a^2)}{(a^2 - x^2)^2} = \frac{4a^2x}{(a^2 - x^2)^2}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du} (u^{-1/2})$$

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{a^2 + x^2}{a^2 - x^2} \right)^{-1/2}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 + x^2}{a^2 - x^2} \right)^{-1/2} \cdot \frac{4a^2x}{(a^2 - x^2)^2}$$

$$\frac{dy}{dx} = \frac{(a^2 + x^2)^{-1/2}}{(a^2 - x^2)^{-1/2}} \cdot \frac{2a^2x}{(a^2 - x^2)^2}$$

$$\frac{dy}{dx} = \frac{2a^2x}{(a^2 + x^2)^{1/2} (a^2 - x^2)^{-1/2+2}}$$

$$\boxed{\frac{dy}{dx} = \frac{2a^2x}{\sqrt{a^2 + x^2} (a^2 - x^2)^{3/2}}} \quad \text{Ans.}$$

Q.2 Find $\frac{dy}{dx}$ if:

(i) $3x + 4y + 7 = 0$

(ii) $xy + y^2 = 2$

(iii) $x^2 - 4xy - 5y = 0$

(iv) $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

(vi) $y(x^2 - 1) = x\sqrt{x^2 + 4}$

Solution:

(i) $3x + 4y + 7 = 0$

Diff. w.r.t 'x'.

$$3 \frac{d}{dx}(x) + 4 \frac{dy}{dx} + \frac{d}{dx}(7) = 0$$

$$3.1 + 4 \frac{dy}{dx} = 0$$

$$4 \frac{dy}{dx} = -3$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{4}}$$

 Ans.

(ii) $xy + y^2 = 2$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(2)$$

$$x \frac{dy}{dx} + y \frac{d}{dx}(x) + 2y \frac{dy}{dx} = 0$$

$$(x + 2y) \frac{dy}{dx} = -y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x + 2y}} \quad \text{Ans.}$$

(iii) $x^2 - 4xy - 5y = 0$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(x^2) - 4 \frac{d}{dx}(xy) - 5 \frac{dy}{dx} = 0$$

$$\begin{aligned}
 2x - 4 \left[x \frac{dy}{dx} + y \cdot 1 \right] - 5 \frac{dy}{dx} &= 0 \\
 2x - 4x \frac{dy}{dx} - 4y - 5 \frac{dy}{dx} &= 0 \\
 -(4x + 5) \frac{dy}{dx} &= -2x + 4y \\
 \frac{dy}{dx} &= \frac{-2(x - 2y)}{-(4x + 5)} \\
 \boxed{\frac{dy}{dx} = \frac{2(x - 2y)}{4x + 5}} &\quad \text{Ans.}
 \end{aligned}$$

(iv) $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Diff. w.r.t 'x'

$$\begin{aligned}
 4 \frac{d}{dx}(x^2) + 2h \frac{d}{dx}(xy) + b \frac{d}{dx}(y^2) + 2g \frac{d}{dx}(x) + 2f \frac{d}{dx}(y) + \frac{d}{dx}(c) &= 0 \\
 4(2x) + 2h \left[x \frac{dy}{dx} + y \cdot 1 \right] + b \cdot 2y \frac{dy}{dx} + 2g \cdot 1 + 2f \frac{dy}{dx} + 0 &= 0 \\
 8x + 2hx \frac{dy}{dx} + 2hy + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} &= 0 \\
 2(hx + by + f) \frac{dy}{dx} &= -8x - 2hy - 2g \\
 \frac{dy}{dx} &= \frac{-2(4x + hy + g)}{2(hx + by + f)} \\
 \boxed{\frac{dy}{dx} = -\frac{4x + hy + g}{hx + by + f}} &\quad \text{Ans.}
 \end{aligned}$$

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$ (L.B 2007) (G.B 2007)

Diff. w.r.t 'x'.

$$\begin{aligned}
 \frac{d}{dx}(x\sqrt{1+y}) + \frac{d}{dx}(y\sqrt{1+x}) &= 0 \\
 x \frac{d}{dx}(\sqrt{1+y}) + \sqrt{1+y} \frac{d}{dx}(x) + y \frac{d}{dx}(\sqrt{1+x}) + \sqrt{1+x} \frac{dy}{dx} &= 0 \\
 x \cdot \frac{1}{2}(1+y)^{-1/2} \frac{dy}{dx} + \sqrt{1+y} \cdot 1 + y \cdot \frac{1}{2}(1+x)^{-1/2} \cdot 1 + \sqrt{1+x} \frac{dy}{dx} &= 0 \\
 \left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x} \right) \frac{dy}{dx} + \sqrt{1+y} + \frac{y}{2\sqrt{1+x}} &= 0 \\
 \left(\frac{x + 2\sqrt{1+x}\sqrt{1+y}}{2\sqrt{1+y}} \right) \frac{dy}{dx} &= -\sqrt{1+y} + \frac{y}{2\sqrt{1+x}}
 \end{aligned}$$

$$\frac{dy}{dx} = \frac{2\sqrt{1+y}}{x+2\sqrt{1+x}\sqrt{1+y}} \left[\frac{-2\sqrt{1+x}\sqrt{1+y}-y}{2\sqrt{1+x}} \right]$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+y}(y+2\sqrt{1+x}\sqrt{1+y})}{\sqrt{1+x}(x+2\sqrt{1+x}\sqrt{1+y})} \text{ Ans.}$$

(vi) $y(x^2 - 1) = x\sqrt{x^2 + 4}$

Diff. w.r.t 'x'.

$$\frac{d}{dx}[y(x^2 - 1)] = \frac{d}{dx}(x\sqrt{x^2 + 4})$$

$$y \frac{d}{dx}(x^2 - 1) + (x^2 - 1) \frac{dy}{dx} = x \frac{d}{dx}(\sqrt{x^2 + 4}) + \sqrt{x^2 + 4} \frac{d}{dx}(x)$$

$$y \cdot 2x + (x^2 - 1) \frac{dy}{dx} = x \frac{1}{2}(x^2 + 4)^{-1/2} \cdot 2x + \sqrt{x^2 + 4}$$

$$(x^2 - 1) \frac{dy}{dx} = \frac{x^2}{\sqrt{x^2 + 4}} + \sqrt{x^2 + 4} - 2xy$$

$$\frac{dy}{dx} = \frac{1}{x^2 - 1} \left[\frac{x^2 + x^2 + 4 - 2xy\sqrt{x^2 + 4}}{\sqrt{x^2 + 4}} \right]$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - 2xy\sqrt{x^2 + 4}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - 2x\sqrt{x^2 + 4} \cdot \frac{x\sqrt{x^2 + 4}}{x^2 - 1}}{(x^2 - 1)\sqrt{x^2 + 4}} \quad \therefore y = \frac{x\sqrt{x^2 + 4}}{x^2 - 1}$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - \frac{2x^2(x^2 + 4)}{x^2 - 1}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{\frac{2x^2(x^2 - 1) + 4(x^2 - 1) - 2x^2(x^2 + 4)}{x^2 - 1}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{2x^4 - 2x^2 + 4x^2 - 4 - 2x^4 - 8x^2}{(x^2 - 1)^2\sqrt{x^2 + 4}}$$

$$= \frac{-6x^2 - 4}{(x^2 - 1)^2\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{-2(3x^2 + 2)}{(x^2 - 1)^2\sqrt{x^2 + 4}}$$

Ans.

Q.3 Find $\frac{dy}{dx}$ of the following parametric functions.

$$(i) \quad x = \theta + \frac{1}{\theta} \quad \text{and} \quad y = \theta + 1 \quad (ii) \quad x = \frac{a(1-t^2)}{1+t^2} \quad \text{and} \quad y = \frac{2bt}{1+t^2}$$

Solution:

$$(i) \quad x = \theta + \frac{1}{\theta} \quad \text{and} \quad y = \theta + 1$$

$$\begin{aligned} x &= \theta + \theta^{-1} \\ \text{Diff. w.r.t. '}\theta\text{'} \\ \frac{dx}{d\theta} &= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1}) \\ \frac{dx}{d\theta} &= 1 + (-1)\theta^{-2} \\ \frac{dx}{d\theta} &= 1 - \frac{1}{\theta^2} \\ \frac{dx}{d\theta} &= \frac{\theta^2 - 1}{\theta^2} \end{aligned}$$

$$\begin{aligned} y &= \theta + 1 \\ \text{Diff. w.r.t. '}\theta\text{'} \\ \frac{dy}{d\theta} &= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(1) \\ \frac{dy}{d\theta} &= 1 + 0 = 1 \end{aligned}$$

By using chain rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \Rightarrow \frac{dy}{dx} = 1 \cdot \frac{\theta^2}{\theta^2 - 1} \\ \boxed{\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}} &\quad \text{Ans.} \end{aligned}$$

$$(ii) \quad x = \frac{a(1-t^2)}{1+t^2} \quad \text{and} \quad y = \frac{2bt}{1+t^2}$$

$$x = \frac{a(1-t^2)}{1+t^2}$$

$$y = \frac{2bt}{1+t^2}$$

Diff. w.r.t. 't'

Diff. w.r.t. 't'

$$\frac{dx}{dt} = a \frac{d}{dt} \left(\frac{1-t^2}{1+t^2} \right)$$

$$\frac{dy}{dt} = 2b \frac{d}{dt} \left[\frac{t}{1+t^2} \right]$$

$$\frac{dx}{dt} = a \left[\frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right]$$

$$\frac{dy}{dt} = 2b \left[\frac{(1+t^2) \frac{d}{dt}(t) - t \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right]$$

$$\frac{dx}{dt} = a \left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} \right]$$

$$\frac{dy}{dt} = 2b \left[\frac{(1+t^2) - t \cdot 2t}{(1+t^2)^2} \right]$$

$$\frac{dx}{dt} = \frac{a 2t(-1-t^2 - 1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b(1+t^2 - 2t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2at(-2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at}$$

$$\frac{dy}{dx} = -\frac{b(1-t^2)}{2at} \quad \text{Ans.}$$

Q.4: Prove that $y \frac{dy}{dx} + x = 0$ if $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ (Guj. Board 2005, 2008)

Solution:

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

$$x = \frac{1-t^2}{1+t^2}$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = \frac{d}{dt} \left(\frac{1-t^2}{1+t^2} \right)$$

$$\frac{dx}{dt} = \frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2t(-1-t^2-1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2t(-2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

$$y = \frac{2t}{1+t^2}$$

Diff. w.r.t 't'

$$\frac{dy}{dt} = 2 \frac{d}{dt} \left(\frac{t}{1+t^2} \right)$$

$$\begin{aligned}\frac{dy}{dt} &= 2 \left[\frac{(1+t^2) \frac{d}{dt}(t) - t \frac{d}{dt}(+t^2)}{(1+t^2)^2} \right] \\ \frac{dy}{dt} &= 2 \left[\frac{(1+t^2) \cdot 1 - t \cdot 2t}{(1+t^2)^2} \right] \\ \frac{dy}{dt} &= \frac{2(1+t^2 - 2t^2)}{(1+t^2)^2} \\ \frac{dy}{dt} &= \frac{2(1-t^2)}{(1+t^2)^2}\end{aligned}$$

By using chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ \frac{dy}{dx} &= \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)}{-4t} \\ \frac{dy}{dx} &= \frac{-(1-t^2)}{2t} = \frac{t^2-1}{2t}\end{aligned}$$

Taking

$$\begin{aligned}y \frac{dy}{dx} + x &= \frac{2t}{1+t^2} \left(\frac{t^2-1}{2t} \right) + \frac{1-t^2}{1+t^2} \\ &= \frac{t^2-1}{1+t^2} + \frac{1-t^2}{1+t^2} \\ &= \frac{t^2-1+1-t^2}{1+t^2} \\ &= \frac{0}{1+t^2} = 0 \quad \text{Hence proved.}\end{aligned}$$

Q.5: Differentiate

(i) $x^2 - \frac{1}{x^2}$ w.r.t. x^4

(ii) $(1+x^2)^n$ w.r.t. x^2

(iii) $\frac{x^2+1}{x^2-1}$ w.r.t. $\frac{x-1}{x+1}$

(iv) $\frac{ax+b}{cx+d}$ w.r.t. $\frac{ax^2+b}{ax^2+d}$

(v) $\frac{x^2+1}{x^2-1}$ w.r.t. x^3

Solution:

(i) $x^2 - \frac{1}{x^2}$ w.r.t. x^4 (L.B 2006)

Let $y = x^2 - \frac{1}{x^2}$, $u = x^4$

$y = x^2 - x^{-2}$
Diff. w.r.t. 'x'

$u = x^4$
Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= 2x - (-2)x^{-3} \\
 &= 2x + \frac{2}{x^3} \\
 &= \frac{2x^4 + 2}{x^3} \\
 &= \frac{2(x^4 + 1)}{x^3}
 \end{aligned}
 \quad \left| \quad \begin{aligned}
 \frac{du}{dx} &= 4x^3
 \end{aligned} \right.$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 \frac{dy}{du} &= \frac{2(x^4 + 1)}{x^3} \times \frac{1}{4x^3} \\
 \frac{dy}{du} &= \frac{x^4 + 1}{2x^6} \quad \text{Ans}
 \end{aligned}$$

(ii) $(1 + x^2)^n$ w.r.t. x^2

Let

$$\begin{aligned}
 y &= (1 + x^2)^n, & u &= x^2 \\
 y &= (1 + x^2)^n & u &= x^4 \\
 \text{Diff. w.r.t. 'x'} & & \text{Diff. w.r.t. 'x'}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (1 + x^2)^n \\
 &= n(1 + x^2)^{n-1} \cdot 2x \\
 &= 2nx(1 + x^2)^{n-1}
 \end{aligned}
 \quad \left| \quad \begin{aligned}
 \frac{du}{dx} &= \frac{d}{dx} x^2 \\
 \frac{du}{dx} &= 2x
 \end{aligned} \right.$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 &= 2nx(1 + x^2)^{n-1} \cdot \frac{1}{2x}
 \end{aligned}$$

$$\frac{dy}{du} = n(1 + x^2)^{n-1}$$

Ans.

(iii) $\frac{x^2 + 1}{x^2 - 1}$ w.r.t. $\frac{x-1}{x+1}$

Let $y = \frac{x^2 + 1}{x^2 - 1}$, $u = \frac{x-1}{x+1}$

$$y = \frac{x^2 + 1}{x^2 - 1}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2 + 1}{x^2 - 1} \right)$$

$$\begin{aligned} &= \frac{(x^2 - 1) \frac{d}{dx}(x^2 + 1) - (x^2 + 1) \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2} \\ &= \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 - 1)^2} \\ &= \frac{2x(-2)}{(x^2 - 1)^2} \end{aligned}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2 - 1)^2}$$

By using chain rule

$$\begin{aligned} \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\ &= \frac{-4x}{(x^2 - 1)^2} \times \frac{(x+1)^2}{2} \\ &= \frac{-2x(x+1)^2}{[(x+1)(x-1)]^2} \\ &= \frac{-2x(x+1)^2}{(x+1)^2(x-1)^2} \end{aligned}$$

$$\boxed{\frac{dy}{du} = \frac{-2x}{(x-1)^2}} \quad \text{Ans.}$$

$$(iv) \quad \frac{ax+b}{cx+d} \text{ w.r.t. } \frac{ax^2+b}{ax^2+d}$$

$$\text{Let } y = \frac{ax+b}{cx+d}, \quad u = \frac{ax^2+b}{ax^2+d}$$

$$y = \frac{ax+b}{cx+d}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{ax+b}{cx+d} \right)$$

$$u = \frac{x-1}{x+1}$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{x-1}{x+1} \right)$$

$$\begin{aligned} &= \frac{(x+1) \frac{d}{dx}(x-1) - (x-1) \frac{d}{dx}(x+1)}{(x+1)^2} \\ &= \frac{x+1-x+1}{(x+1)^2} \end{aligned}$$

$$\frac{du}{dx} = \frac{2x}{(x+1)^2}$$

$$u = \frac{ax^2+b}{ax^2+d}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{ax^2+b}{ax^2+d} \right)$$

$$\begin{aligned}
 &= \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2} \\
 &= \frac{(cx+d)a - (ax+b)c}{(cx+d)^2} \\
 &= \frac{acx + ad - acx - bc}{(cx+d)^2} \\
 \frac{dy}{dx} &= \frac{ad - bc}{(cx+d)^2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(ax^2 + d)\frac{d}{dx}(ax^2 + b) - (ax^2 + b)\frac{d}{dx}(ax^2 + d)}{(ax^2 + d)^2} \\
 &= \frac{(ax^2 + d)(2ax) - (ax^2 + b)(2ax)}{(ax^2 + d)^2} \\
 \frac{du}{dx} &= \frac{2ax(ax^2 + d - ax^2 - b)}{(ax^2 + d)^2} \\
 \frac{du}{dx} &= \frac{2ax(d - b)}{(ax^2 + d)^2}
 \end{aligned}$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 &= \frac{ad - bc}{(cx+d)^2} \cdot \frac{(ax^2 + d)^2}{2ax(d - b)}
 \end{aligned}$$

$$\boxed{\frac{dy}{du} = \frac{(ad - bc)(ax^2 + d)^2}{2ax(cx+d)^2(d - b)}}$$

Ans.

(v) $\frac{x^2 + 1}{x^2 - 1}$ w.r.t. x^3 (G.B 2003)

Let

$$y = \frac{x^2 + 1}{x^2 - 1}$$

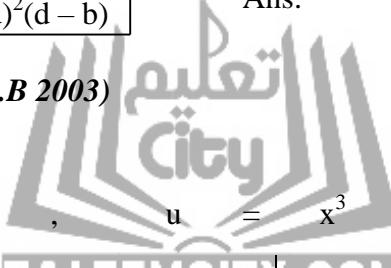
$$y = \frac{x^2 + 1}{x^2 - 1}$$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x^2 + 1}{x^2 - 1} \right) \\
 &= \frac{(x^2 - 1)\frac{d}{dx}(x^2 + 1) - (x^2 + 1)\frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2} \\
 &= \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2} \\
 &= \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 - 1)^2}
 \end{aligned}$$

$$\frac{dy}{dx} = \frac{2x(-2)}{(x^2 - 1)^2}$$

$$= \frac{-4x}{(x^2 - 1)^2}$$



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$$u = x^3$$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{du}{dx} &= \frac{d}{dx} (x^3) \\
 &= 3x^2
 \end{aligned}$$

By using chain rule

$$\begin{aligned}\frac{dx}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\ \frac{dx}{du} &= \frac{-4x}{(x^2 - 1)^2} \times \frac{1}{3x^2} \\ \frac{dx}{du} &= \frac{-4x}{3x(x^2 - 1)^2} \quad \text{Ans.}\end{aligned}$$

EXERCISE 2.5

Q.1: Differentiate the following trigonometric functions from the first principles.

- | | | |
|-----------------------|----------------|---------------------------|
| (i) $\sin 2x$ | (ii) $\tan 3x$ | (iii) $\sin 2x + \cos 2x$ |
| (iv) $\cos x^2$ | (v) $\tan^2 x$ | (vi) $\sqrt{\tan x}$ |
| (vii) $\cos \sqrt{x}$ | | |

Solution:

(i) $\sin 2x \quad (\text{L.B 2003})$

Let $y = \sin 2x$

$$y + \delta y = \sin 2(x + \delta x)$$

$$\delta y = \sin(2x + 2\delta x) - y$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x$$

$$\delta y = 2 \cos\left(\frac{2x + 2\delta x + 2x}{2}\right) \cdot \sin\left(\frac{2x + 2\delta x - 2x}{2}\right) \quad \because y = \sin 2x$$

$$[\because \sin p - \sin q = 2 \cos\left(\frac{p+q}{2}\right) \sin\left(\frac{p-q}{2}\right)]$$

$$\delta y = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin\left(\frac{2\delta x}{2}\right)$$

$$\delta y = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin(\delta x)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \frac{\sin \delta x}{\delta x}$$