

By using chain rule

$$\begin{aligned}\frac{dx}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\ \frac{dx}{du} &= \frac{-4x}{(x^2 - 1)^2} \times \frac{1}{3x^2} \\ \frac{dx}{du} &= \frac{-4x}{3x(x^2 - 1)^2} \quad \text{Ans.}\end{aligned}$$

### EXERCISE 2.5

**Q.1:** Differentiate the following trigonometric functions from the first principles.

- |                       |                |                           |
|-----------------------|----------------|---------------------------|
| (i) $\sin 2x$         | (ii) $\tan 3x$ | (iii) $\sin 2x + \cos 2x$ |
| (iv) $\cos x^2$       | (v) $\tan^2 x$ | (vi) $\sqrt{\tan x}$      |
| (vii) $\cos \sqrt{x}$ |                |                           |

**Solution:**

(i)  $\sin 2x \quad (\text{L.B 2003})$

Let  $y = \sin 2x$

$$y + \delta y = \sin 2(x + \delta x)$$

$$\delta y = \sin(2x + 2\delta x) - y$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x$$

$$\delta y = 2 \cos\left(\frac{2x + 2\delta x + 2x}{2}\right) \cdot \sin\left(\frac{2x + 2\delta x - 2x}{2}\right) \quad \because y = \sin 2x$$

$$[\because \sin p - \sin q = 2 \cos\left(\frac{p+q}{2}\right) \sin\left(\frac{p-q}{2}\right)]$$

$$\delta y = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin\left(\frac{2\delta x}{2}\right)$$

$$\delta y = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin(\delta x)$$

Dividing both sides by  $\delta x$

$$\frac{\delta y}{\delta x} = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit  $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \frac{\sin \delta x}{\delta x}$$

$$\frac{dy}{dx} = 2 \cos\left(\frac{4x}{2}\right) \cdot 1$$

$$\boxed{\frac{dy}{dx} (\sin 2x) = 2 \cos 2x} \quad \text{Ans.}$$

**(ii) tan 3x**

Let  $y = \tan 3x$

$$y + \delta y = \tan 3(x + \delta x)$$

$$\delta y = \tan(3x + 3\delta x) - y$$

$$\delta y = \tan(3x + 3\delta x) - \tan 3x \quad \therefore y = \tan 3x$$

$$\delta y = \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$\delta y = \frac{\sin(3x + 3\delta x) \cos 3x - \cos(3x + 3\delta x) \sin 3x}{\cos(3x + 3\delta x) \cos 3x}$$

$$\delta y = \frac{\sin(3x + 3\delta x - 3x)}{\cos(3x + 3\delta x) \cos 3x} \quad [\because \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta]$$

$$\delta y = \frac{\sin 3\delta x}{\cos(3x + 3\delta x) \cos 3x}$$

Dividing both sides by  $\delta x$

$$\frac{\delta y}{\delta x} = \frac{3}{\cos(3x + 3\delta x) \cos 3x} \cdot \frac{\sin 3\delta x}{3\delta x}$$

Taking limit  $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{3}{\cos(3x + 3\delta x) \cos 3x} \cdot \frac{\sin 3\delta x}{3\delta x}$$

$$\frac{dy}{dx} = \frac{3}{\cos 3x \cdot \cos 3x} \cdot 1$$

$$\frac{dy}{dx} = \frac{3}{\cos^2 3x}$$

$$\boxed{\frac{d}{dx} (\tan 3x) = 3 \sec^2 3x} \quad \text{Ans.}$$

**(iii) sin 2x + cos 2x**

Let  $y = \sin 2x + \cos 2x$

$$y + \delta y = \sin 2(x + \delta x) + \cos 2(x + \delta x)$$

$$\delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - y$$

$$\delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - (\sin 2x + \cos 2x)$$

$$\therefore y = \sin 2x + \cos 2x$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x + \cos(2x + 2\delta x) - \cos 2x$$

$$\begin{aligned}\delta y &= 2 \cos\left(\frac{2x + 2\delta x + 2x}{2}\right) \cdot \sin\left(\frac{2x + 2\delta x - 2x}{2}\right) - 2 \sin\left(\frac{2x + 2\delta x + 2x}{2}\right) \\ &\quad \sin\left(\frac{2x + 2\delta x - 2x}{2}\right)\end{aligned}$$

$$\delta y = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin\left(\frac{2\delta x}{2}\right) - 2 \sin\left(\frac{4x + 2\delta x}{2}\right) \sin\left(\frac{2\delta x}{2}\right)$$

$$\delta y = 2 \sin \delta x \left[ \cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right) \right]$$

Dividing both sides by  $\delta x$

$$\frac{\delta y}{\delta x} = \frac{2 \sin \delta x}{\delta x} \left[ \cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right) \right]$$

Taking limit  $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 2 \frac{\sin \delta x}{\delta x} \left[ \cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right) \right]$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \left[ \cos\left(\frac{4x}{2}\right) - \sin\left(\frac{4x}{2}\right) \right]$$

$$\frac{d}{dx} (\sin 2x + \cos 2x) = 2(\cos 2x - \sin 2x)$$

Ans.

(iv)  **$\cos x^2$  (L.B 2003)**

$$\text{Let } y = \cos x^2$$

$$y + \delta y = \cos(x + \delta x)^2$$

$$\delta y = \cos(x + \delta x)^2 - y$$

$$\delta y = \cos(x + \delta x)^2 - \cos x^2 \quad \because (y = \cos x^2)$$

$$\delta y = -2 \sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \sin\left[\frac{(x + \delta x)^2 - x^2}{2}\right]$$

$$\delta y = -2 \sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \sin\left[\frac{x^2 + \delta x^2 + 2x\delta x - x^2}{2}\right]$$

$$\delta y = -2 \sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \sin\left[\frac{\delta x(\delta x + 2x)}{2}\right]$$

Dividing both sides by  $\delta x$

$$\frac{\delta y}{\delta x} = -2 \sin \left[ \frac{(x + \delta x)^2 + x^2}{2} \right] \cdot \frac{\sin \left[ \frac{\delta x (\delta x + 2x)}{2} \right]}{\left( \frac{2}{\delta x + 2x} \right) \cdot \delta x \left( \frac{\delta x + 2x}{2} \right)}$$

Taking limit  $\delta x \rightarrow 0$

$$\begin{aligned}\frac{dy}{dx} &= -2 \sin \left( \frac{x^2 + x^2}{2} \right) \cdot \frac{1}{2x} \cdot 1 \\ &= -2 \sin \left( \frac{2x^2}{2} \right) \cdot x\end{aligned}$$

$$\boxed{\frac{d}{dx}(\cos x^2) = -2x \sin x^2} \quad \text{Ans.}$$

(v)  $\tan^2 x$

Let

$$\begin{aligned}y &= \tan^2 x \\ y + \delta y &= \tan^2(x + \delta x) \\ \delta y &= \tan^2(x + \delta x) - y \\ \delta y &= \tan^2(x + \delta x) - \tan^2 x \quad \therefore y = \tan^2 x \\ \delta y &= [\tan(x + \delta x) + \tan x] [\tan(x + \delta x) - \tan x] \\ \delta y &= [\tan(x + \delta x) + \tan x] \left[ \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right] \\ \delta y &= [\tan(x + \delta x) + \tan x] \left[ \frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \sin x}{\cos(x + \delta x) \cos x} \right] \\ \delta y &= \left[ \frac{\tan(x + \delta x) + \tan x}{\cos(x + \delta x) \cdot \cos x} \right] \sin(x + \delta x - x)\end{aligned}$$

Dividing both sides by  $\delta x$ .

$$\frac{\delta y}{\delta x} = \frac{[\tan(x + \delta x) + \tan x]}{\cos(x + \delta x) \cdot \cos x} \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit  $\delta x \rightarrow 0$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\tan x + \tan x}{\cos x \cdot \cos x} \cdot 1 \\ \frac{dy}{dx} &= \frac{2 \tan x}{\cos^2 x}\end{aligned}$$

$$\boxed{\frac{d}{dx}(\tan^2 x) = 2 \tan x \sec^2 x} \quad \text{Ans.}$$

(vi)  $\sqrt{\tan x}$  (L.B 2003, 2004)

Let

$$\begin{aligned}
 y &= \sqrt{\tan x} \\
 y + \delta y &= \sqrt{\tan(x + \delta x)} \\
 \delta y &= \sqrt{\tan(x + \delta x)} - y \\
 \delta y &= \sqrt{\tan(x + \delta x)} - \sqrt{\tan x} \quad (\because y = \sqrt{\tan x}) \\
 \delta y &= [\sqrt{\tan(x + \delta x)} - \sqrt{\tan x}] \left[ \frac{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \right] \\
 \delta y &= \frac{(\sqrt{\tan(x + \delta x)})^2 - (\sqrt{\tan x})^2}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \\
 \delta y &= \frac{\tan(x + \delta x) - \tan x}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \\
 \delta y &= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[ \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right] \\
 \delta y &= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[ \frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \cdot \sin x}{\cos(x + \delta x) \cdot \cos x} \right] \\
 \delta y &= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \times \frac{\sin(x + \delta x - x)}{\cos(x + \delta x) \cdot \cos x} \\
 \delta y &= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \times \frac{\sin \delta x}{\cos(x + \delta x) \cdot \cos x}
 \end{aligned}$$

Dividing both sides by  $\delta x$

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \cdot \frac{1}{\cos(x + \delta x) \cdot \cos x} \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit  $\delta x \rightarrow 0$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}} \cdot \frac{1}{\cos x \cdot \cos x} \cdot 1$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan x}} \cdot \frac{1}{\cos^2 x}$$

$\frac{d}{dx} (\sqrt{\tan x})$	$= \frac{\sec^2 x}{2\sqrt{\tan x}}$	Ans.
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(vii)  $\cos \sqrt{x}$  (L.B 2004)

Let

$$\begin{aligned}
 y &= \cos \sqrt{x} \\
 y + \delta y &= \cos \sqrt{x + \delta x} \\
 \delta y &= \cos \sqrt{x + \delta x} - y \\
 \delta y &= \cos \sqrt{x + \delta x} - \cos \sqrt{x} && \therefore y = \cos \sqrt{x} \\
 \delta y &= -2 \sin \left( \frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \cdot \sin \left( \frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right) \\
 \delta y &= -2 \sin \left( \frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \cdot \frac{\sin \left( \frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{\delta x} \\
 \delta y &= -2 \sin \left( \frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \cdot \frac{\sin \left( \frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{2 \left( \sqrt{x + \delta x} + \sqrt{x} \right) \frac{(\sqrt{x + \delta x} - \sqrt{x})}{2}} \\
 &\therefore \delta x = (\sqrt{x + \delta x} + \sqrt{x})(\sqrt{x + \delta x} - \sqrt{x})
 \end{aligned}$$

Taking limit  $\delta x \rightarrow 0$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{-\sin \left( \frac{\sqrt{x} + \sqrt{x}}{2} \right)}{\sqrt{x} + \sqrt{x}} \cdot 1 \\
 \frac{dy}{dx} &= \frac{-\sin \left( \frac{2\sqrt{x}}{2} \right)}{2\sqrt{x}}
 \end{aligned}$$

$$\frac{d}{dx} (\cos \sqrt{x}) = \frac{-\sin \sqrt{x}}{2\sqrt{x}}$$

Ans.

**Q.2:** Differentiate the following w.r.t. the variable involved.

- |   |                                      |
|---|--------------------------------------|
| (i) $x^2 \sec 4x$                       | (ii) $\tan^3 \theta \sec^2 \theta$   |
| (iii) $(\sin 2\theta - \cos 3\theta)^2$ | (iv) $\cos \sqrt{x} + \sqrt{\sin x}$ |

**Solution:**

- (i)  $x^2 \sec 4x$  (G.B 2005)

Let  $y = x^2 \sec 4x$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (x^2 \sec 4x) \\
 &= x^2 \frac{d}{dx} (\sec 4x) + \sec 4x \frac{d}{dx} (x^2) \\
 &= x^2 \sec 4x \tan 4x \cdot 4 + \sec 4x \cdot 2x \\
 &= 2x \sec 4x (1 + 2x \tan 4x) \quad \text{Ans.}
 \end{aligned}$$

(ii)  **$\tan^3\theta \sec^2\theta$** 

Let  $y = \tan^3\theta \sec^2\theta$

Diff. w.r.t. ' $\theta$ '

$$\begin{aligned}
 \frac{dy}{d\theta} &= \frac{d}{d\theta} (\tan^3\theta \sec^2\theta) \\
 &= \tan^3\theta \frac{d}{d\theta} (\sec^2\theta) + \sec^2\theta \frac{d}{d\theta} (\tan^3\theta) \\
 &= \tan^3\theta \cdot 2\sec\theta \cdot \sec\theta \tan\theta + \sec^2\theta \cdot 3\tan^2\theta \cdot \sec^2\theta \\
 &= \tan^2\theta \sec^2\theta (2\tan^2\theta + 3\sec^2\theta) \quad \text{Ans.}
 \end{aligned}$$

(iii)  **$(\sin 2\theta - \cos 3\theta)^2$** 

Let  $y = (\sin 2\theta - \cos 3\theta)^2$

Diff. w.r.t. ' $\theta$ '

$$\begin{aligned}
 \frac{dy}{d\theta} &= \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta)^2 \\
 \frac{dy}{d\theta} &= 2(\sin 2\theta - \cos 3\theta) \cdot \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta) \\
 \frac{dy}{d\theta} &= 2(\sin 2\theta - \cos 3\theta) (2\cos 2\theta + 3\sin 3\theta) \quad \text{Ans.}
 \end{aligned}$$

(iv)  **$\cos \sqrt{x} + \sqrt{\sin x}$  (L.B 2008)**

Let  $y = \cos \sqrt{x} + \sqrt{\sin x}$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (\cos \sqrt{x}) + \frac{d}{dx} (\sqrt{\sin x}) \\
 &= -\sin \sqrt{x} \frac{d}{dx} (\sqrt{x}) + \frac{1}{2} (\sin x)^{-\frac{1}{2}} \cdot \cos x \\
 &= \frac{-\sin \sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}} \\
 &= \frac{1}{2} \left( \frac{-\sin \sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}} \right) \quad \text{Ans.}
 \end{aligned}$$

**Q.3:** Find  $\frac{dy}{dx}$  if

(i)  $y = x \cos y$  (L.B 2009)      (ii)  $x = y \sin y$  (L.B 2009)

**Solution:**

(i)  $y = x \cos y$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} (x \cos y)$$

$$\frac{dy}{dx} = x \frac{d}{dx} (\cos y) + \cos y \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = x \times (-\sin y \frac{dy}{dx}) + \cos y \cdot 1$$

$$\frac{dy}{dx} + x \sin y \frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx} (1 + x \sin y) = \cos y$$

$$\boxed{\frac{dy}{dx} = \frac{\cos y}{1 + x \sin y}}$$

Ans.

(ii)  $x = y \sin y$

Diff. w.r.t. 'x'

$$\frac{d}{dx} (x) = \frac{d}{dx} (y \sin y)$$

$$1 = y \frac{d}{dx} (\sin y) + \sin y \frac{dy}{dx}$$

$$1 = y \cos y \frac{dy}{dx} + \sin y \frac{dy}{dx}$$

$$1 = (y \cos y + \sin y) \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y \cos y + \sin y}}$$

Ans.

**Q.4:** Find the derivative w.r.t. 'x'

(i)  $\cos \sqrt{\frac{1+x}{1+2x}}$

(ii)  $\sin \sqrt{\frac{1+2x}{1+x}}$

**Solution:**

(i)  $\cos \sqrt{\frac{1+x}{1+2x}}$  (G.B 2005)

Let

$$y = \cos \sqrt{\frac{1+x}{1+2x}}$$

Diff. w.r.t. 'x'

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left( \cos \sqrt{\frac{1+x}{1+2x}} \right) \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{d}{dx} \left( \sqrt{\frac{1+x}{1+2x}} \right) \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \left( \frac{1+x}{1+2x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left( \frac{1+x}{1+2x} \right) \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \cdot \frac{(1+x)^{-\frac{1}{2}}}{(1+2x)^{\frac{1}{2}}} \left[ \frac{(1+2x) \frac{d}{dx}(1+x) - (1+x) \cdot \frac{d}{dx}(1+2x)}{(1+2x)^2} \right] \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \cdot \frac{[1+2x - (1+x) \cdot 2]}{(1+x)^{\frac{1}{2}} \text{ eq } (1+2x)^{\frac{1}{2}-\frac{1}{2}}} \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+2x - 2 - 2x)}{2 \sqrt{1+x} (1+2x)^{\frac{3}{2}}} \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{-1}{2 \sqrt{1+x} (1+2x)^{\frac{3}{2}}} \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{\sin \sqrt{\frac{1+x}{1+2x}}}{2 \sqrt{1+x} (1+2x)^{\frac{3}{2}}}} \quad \text{Ans.}$$

(ii)  $\sin \sqrt{\frac{1+2x}{1+x}}$  (G.B 2004)

Let  $y = \sin \sqrt{\frac{1+2x}{1+x}}$

Diff. w.r.t. 'x'

$$\begin{aligned} \frac{dy}{dx} &= \left[ \sin \sqrt{\frac{1+2x}{1+x}} \right] \\ &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{d}{dx} \left( \sqrt{\frac{1+2x}{1+x}} \right) \end{aligned}$$

$$\begin{aligned}
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2} \left( \frac{1+2x}{1+x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left( \frac{1+2x}{1+x} \right) \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2} \frac{(1+2x)^{-\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}} \left[ \frac{(1+x) \frac{d}{dx}(1+2x) - (1+2x) \cdot \frac{d}{dx}(1+x)}{(1+x)^2} \right] \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{[(1+x)2 - (1+2x)]}{2(1+2x)^{\frac{1}{2}} (1+2x)^{\frac{-1}{2}+2}} \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{(2+2x-1-2x)}{2\sqrt{1+2x} (1+x)^{\frac{3}{2}}}
 \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{\cos \sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x} (1+x)^{\frac{3}{2}}}}$$

Ans.

**Q.5: Differentiate**(i)  $\sin x$  w.r.t.  $\cot x$ (ii)  $\sin^2 x$  w.r.t.  $\cos^4 x$ **Solution:** $\sin x$  w.r.t.  $\cot x$  (L.B 2009)

Let

$$y = \sin x, u = \cot x$$

$$y = \sin x$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \cos x$$

$$u = \cot x$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = -\operatorname{cosec}^2 x$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 &= \cos x \cdot \frac{-1}{\operatorname{cosec}^2 x}
 \end{aligned}$$

$$\boxed{\frac{dy}{du} = -\cos x \sin^2 x}$$

Ans.

(ii)  $\sin^2 x$  w.r.t.  $\cos^4 x$  (G.B 2003, L.B 2009, L.B 2004, L.B 2008)

Let

$$y = \sin^2 x, u = \cos^4 x$$

$y = \sin^2 x$ Diff. w.r.t. 'x' $\frac{dy}{dx} = 2 \sin x \cos x$	$u = \cos^4 x$ Diff. w.r.t. 'x' $\frac{du}{dx} = 4 \cos^3 x - \sin x$ $= -4 \sin x \cos^3 x$
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By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 &= 2 \sin x \cos x \cdot \frac{-1}{4 \sin x \cos^3 x} \\
 &= \frac{-1}{2 \cos^2 x} = \frac{-1}{2} \sec^2 x \quad \text{Ans.}
 \end{aligned}$$

**Q.6:** If  $\tan y (1 + \tan x) = 1 - \tan x$ , show that  $\frac{dy}{dx} = -1$  (G.B 2009)

**Solution:**

$$\tan y (1 + \tan x) = 1 - \tan x$$

$$\tan y = \frac{1 - \tan x}{1 + \tan x}$$

$$\tan y = \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x}$$

$$\tan y = \tan \left( \frac{\pi}{4} - x \right)$$

$$y = \frac{\pi}{4} - x$$

Diff. w.r.t. 'x'

$\frac{dy}{dx} = -1$
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Hence proved.

**Q.7:** If  $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}} + \dots \dots \infty$ , prove that  $(2y - 1) \frac{dy}{dx} = \sec^2 x$ .

**Solution:**

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}} + \dots \dots \infty$$

Squaring on both sides

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x}} + \dots \dots \infty$$

$$y^2 = \tan x + y$$

Diff. w.r.t. 'x'

$$2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$(2y - 1) \frac{dy}{dx} = \sec^2 x$$

Hence proved.

**Q.8:** If  $x = a \cos^3 \theta$ ,  $y = b \sin^3 \theta$ , show that  $a \frac{dy}{dx} + b \tan \theta = 0$

(G.B 2004, G.B 2011, G.B 2007)

**Solution:**

$$x = a \cos^3 \theta$$

Diff. w.r.t. 'θ'

$$\begin{aligned} \frac{dx}{d\theta} &= 3a \cos^2 \theta (-\sin \theta) \\ &= -3a \sin \theta \cos^2 \theta \end{aligned}$$

$$y = b \sin^3 \theta$$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = 3b \sin^2 \theta \cos \theta$$

By using chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \\ &= 3b \sin^2 \theta \cos \theta \cdot \frac{-1}{3a \sin \theta \cos^2 \theta} \\ &= -\frac{b}{a} \tan \theta \end{aligned}$$

$$a \frac{dy}{dx} = -b \tan \theta$$

$$a \frac{dy}{dx} + b \tan \theta = 0$$

Hence proved.

**Q.9:** Find  $\frac{dy}{dx}$  if  $x = a (\cos t + \sin t)$ ,  $y = a (\sin t - t \cos t)$

**Solution:**

$$x = a (\cos t + \sin t)$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = a (-\sin t + \cos t)$$

$$y = a (\sin t - t \cos t)$$

Diff. w.r.t. 't'

$$\frac{dy}{dt} = a [\cos t - \{ t . -\sin t + \cos t . 1 \}]$$

$$\left| \begin{array}{l} \frac{dy}{dt} = a \cos t + a t \cdot \sin t - a \cos t \\ \frac{dy}{dt} = a t \cdot \sin t \end{array} \right.$$

By using chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= at \sin t \cdot \frac{1}{a(-\sin t + \cos t)} \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{t \sin t}{\cos t - \sin t}} \text{ Ans.}$$

**Q.10:** Differentiate w.r.t. 'x'

(i)  $\cos^{-1} \frac{x}{a}$

(ii)  $\cot^{-1} \frac{x}{a}$

(iii)  $\frac{1}{a} \sin^{-1} \frac{a}{x}$

(iv)  $\sin^{-1} \sqrt{1-x^2}$

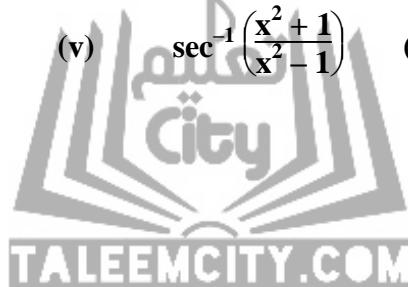
(v)  $\sec^{-1} \left( \frac{x^2+1}{x^2-1} \right)$

(vi)  $\cot^{-1} \left( \frac{2x}{1-x^2} \right)$

(vii)  $\cos^{-1} \left( \frac{1-x^2}{1+x^2} \right)$

**Solution:**

(i)  $\cos^{-1} \frac{x}{a}$

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Let  $y = \cos^{-1} \frac{x}{a}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx} \left( \frac{x}{a} \right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}} \quad \text{Ans.}$$

(ii)  $\cot^{-1} \frac{x}{a}$  (L.B 2006)

Let  $y = \cot^{-1} \frac{x}{a}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx} \left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{1 + \frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{-1}{\frac{a^2 + x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{-a}{a^2 + x^2} \quad \text{Ans.}$$

(iii)  $\frac{1}{a} \sin^{-1} \frac{a}{x}$  (L.B 2010)

Let  $y = \frac{1}{a} \sin^{-1} \frac{a}{x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{a} \frac{1}{\sqrt{1 - \left(\frac{a}{x}\right)^2}} \cdot \frac{d}{dx} \left(\frac{a}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{a} \frac{1}{\sqrt{1 - \frac{a^2}{x^2}}} \cdot a \frac{d}{dx} (x^{-1})$$

$$= \frac{1}{\sqrt{\frac{x^2 - a^2}{x^2}}} \cdot -1 x^{-2}$$



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$$\begin{aligned}
 &= \frac{1}{\sqrt{x^2 - a^2}} \cdot -\frac{1}{x^2} \\
 &= \frac{-1}{x \sqrt{x^2 - a^2}} \text{ Ans.}
 \end{aligned}$$

(iv)  $\sin^{-1} \sqrt{1-x^2}$ 

Let  $y = \sin^{-1} \sqrt{1-x^2}$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{d}{dx} (\sqrt{1-x^2}) \\
 \frac{dy}{dx} &= \frac{1}{\sqrt{1-1+x^2}} \cdot \frac{1}{2} (1-x^2)^{\frac{-1}{2}} \cdot \frac{d}{dx} (1-x^2) \\
 \frac{dy}{dx} &= \frac{1}{\sqrt{x^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-2x) \\
 &= \frac{-x}{x \sqrt{1-x^2}}
 \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

Ans.

(v)  $\sec^{-1} \left( \frac{x^2+1}{x^2-1} \right)$ 

$$\left( \because \frac{d}{dx} \sec^{-1} x = \frac{1}{x \sqrt{x^2-1}} \right)$$

Let

$$y = \sec^{-1} \left( \frac{x^2+1}{x^2-1} \right)$$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{\left( \frac{x^2+1}{x^2-1} \right) \sqrt{\left( \frac{x^2+1}{x^2-1} \right)^2 - 1}} \cdot \frac{d}{dx} \left( \frac{x^2+1}{x^2-1} \right) \\
 &= \frac{x^2-1}{(x^2+1) \sqrt{(x^2+1)^2 - 1}} \left[ \frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(x^2 - 1)2x - (x^2 + 1).2x}{(x^2 + 1)(x^2 - 1)\sqrt{\frac{(x^2 + 1)^2 - (x^2 - 1)^2}{(x^2 - 1)^2}}} \\
 &= \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 + 1)(x^2 - 1)\sqrt{\frac{x^4 + 1 + 2x^2 - (x^4 + 1 - 2x^2)}{x^2 - 1}}} \\
 &= \frac{2x(-2)}{(x^2 + 1)\sqrt{x^4 + 1 + 2x^2 - x^4 - 1 + 2x^2}} \\
 &= \frac{-4x}{(x^2 + 1)\sqrt{4x^2}} \\
 &= \frac{-4x}{(x^2 + 1).2x} = \frac{-2}{x^2 + 1} \quad \text{Ans.}
 \end{aligned}$$

(vi)  $\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Let  $y = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Diff. w.r.t. 'x',  $\left(\frac{d}{dx} \cot^{-1}x = \frac{-1}{1+x^2}\right)$

$$\frac{dy}{dx} = \frac{-1}{1 + \left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{d}{dx} \left(\frac{2x}{1-x^2}\right)$$

$$= \frac{-2}{1 + \frac{4x^2}{(1-x^2)^2}} \left[ \frac{(1-x^2) \frac{d}{dx}(x) - x \frac{d}{dx}(1-x^2)}{(1-x^2)^2} \right]$$

$$= \frac{-2}{\frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2}} \left[ \frac{(1-x^2) - x(-2x)}{(1-x^2)^2} \right]$$

$$= \frac{-2[1-x^2+2x^2]}{1+x^4-2x^2+4x^2}$$

$$= \frac{-2(1+x^2)}{1+x^4+2x^2}$$

$$= \frac{-2(1+x^2)}{(1+x^2)^2}$$

$$= \frac{-2}{1+x^2} \quad \text{Ans.}$$

(vii)  $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \quad (\text{L.B 2004})$

Let  $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Diff. w.r.t. 'x'

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \quad \left(\frac{d}{dx} \cos^{-1}x = \frac{-1}{x\sqrt{1-x^2}}\right) \\ &= \frac{-1}{\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}} \left[ \frac{(1+x^2)\frac{d}{dx}(1-x^2)-(1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2} \right] \\ &= \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}} \left[ \frac{(1+x^2)(-2x)-(1-x^2).2x}{(1+x^2)^2} \right] \\ &= \frac{-2x[-1-x^2-1+x^2]}{\sqrt{1+x^4+2x^2-(1+x^4-2x^2)}(1+x^2)^2} \\ &= \frac{-2x(-2)}{\sqrt{1+x^4+2x^2-1-x^4+2x^2}(1+x^2)} \\ &= \frac{4x}{\sqrt{4x^2}(1+x^2)} \\ &= \frac{4x}{2x(1+x^2)} \\ &= \frac{2}{1+x^2} \quad \text{Ans.} \end{aligned}$$

**Q.11:** Show that  $\frac{dy}{dx} = \frac{y}{x}$  if  $\frac{y}{x} = \tan^{-1}\frac{x}{y}$  (*L.B 2003, G.B 2007, L.B 2007, G.B 2008*)

**Solution:**

$$\frac{y}{x} = \tan^{-1} \frac{x}{y}$$

$$y = x \tan^{-1} \frac{x}{y}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x \frac{d}{dx} \left( \tan^{-1} \frac{x}{y} \right) + \tan^{-1} \left( \frac{x}{y} \right) \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = x \cdot \frac{1}{1 + \left( \frac{x}{y} \right)^2} \cdot \frac{d}{dx} \left( \frac{x}{y} \right) + \tan^{-1} \left( \frac{x}{y} \right)$$

$$\frac{dy}{dx} = \frac{x}{1 + \frac{x^2}{y^2}} \left[ \frac{y \frac{d}{dx}(x) - x \frac{dy}{dx}}{y^2} \right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{1 + \frac{x^2}{y^2}} \left[ \frac{y - x \frac{dy}{dx}}{y^2} \right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{\frac{y^2 + x^2}{y^2}} \left[ \frac{\left( y - \frac{dy}{dx} \right)}{y^2} \right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} - \frac{x^2}{x^2 + y^2} \frac{dy}{dx} + \frac{y}{x}$$

$$\frac{dy}{dx} + \frac{x^2}{x^2 + y^2} \frac{dy}{dx} = y \left[ \frac{x}{x^2 + y^2} + \frac{1}{x} \right]$$

$$\frac{dy}{dx} \left[ 1 + \frac{x}{x^2 + y^2} \right] = y \left[ \frac{x^2 + x^2 + y^2}{x(x^2 + y^2)} \right]$$

$$\frac{dy}{dx} \left( \frac{x^2 + y^2 + x^2}{x^2 + y^2} \right) = y \left[ \frac{2x^2 + y^2}{x(x^2 + y^2)} \right] = \frac{y(x^2 + y^2)(2x^2 + y^2)}{x(x^2 + y^2)(2x^2 + y^2)}$$

$\frac{dy}{dx}$	$=$	$\frac{y}{x}$
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Hence proved.

**Q.12:** If  $y = \tan(P \tan^{-1} x)$ , show that  $(1 + x^2)y_1 - P(1 + y^2) = 0$   
 (L.B 2006) (G.B 2006)

**Solution:**

$$\begin{aligned}\tan^{-1} y &= \tan(P \tan^{-1} x) \\ \tan^{-1} y &= P \tan^{-1} x\end{aligned}$$

Diff. w.r.t. 'x'

$$\frac{1}{1+y^2} \cdot y_1 = P \frac{1}{1+x^2}$$

$$(1+x^2) y_1 = P(1+y^2)$$

$$\boxed{(1+x^2) y_1 - P(1+y^2) = 0} \quad \text{Hence proved.}$$

**EXERCISE 2 . 6****Q.1: Find  $f'(x)$  if**

- |  |   |
|--|---|
| (i) $f(x) = e^{\sqrt{x}-1}$                    | (ii) $f(x) = x^3 e^{\frac{1}{x}} (x \neq 0)$            |
| (iii) $f(x) = ex(1+\ell nx)$                   | (iv) $f(x) = \frac{e^x}{e^{-x}+1}$                      |
| (v) $f(x) = \ell n(e^x + e^{-x})$              | (vi) $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$ |
| (vii) $f(x) = \sqrt{\ell n(e^{2x} + e^{-2x})}$ | (viii) $f(x) = \ell n(\sqrt{e^{2x} + e^{-2x}})$         |

**Solution:**

(i)  $f(x) = e^{\sqrt{x}-1}$

Diff. w.r.t. 'x'

$$f'(x) = e^{\sqrt{x}-1} \cdot \frac{d}{dx} (\sqrt{x}-1)$$

$$f'(x) = e^{\sqrt{x}-1} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$\boxed{f'(x) = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}}} \quad \text{Ans.}$$

(ii)  $f(x) = x^3 e^{\frac{1}{x}} (x \neq 0)$

Diff. w.r.t. 'x'

$$f'(x) = x^3 \frac{d}{dx} \left( e^{\frac{1}{x}} \right) + e^{\frac{1}{x}} \frac{d}{dx} (x^3)$$