

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \frac{d}{dx} (\tanh x)$$

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \sec h^2 x$$

$$\frac{dy}{dx} = \frac{1}{\cosh^2 x}$$

$$\frac{\sinh x}{\cosh x}$$

$$\frac{dy}{dx} = \frac{1}{\sinh x \cosh x}$$

$$\frac{dy}{dx} = \frac{2}{2 \sinh x \cosh x} = \frac{2}{\sinh 2x}$$

$$\frac{dy}{dx} = 2 \operatorname{cosech} 2x$$

Ans.

(vi) $y = \sinh^{-1}\left(\frac{x}{2}\right)$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \frac{x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4+x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{4+x^2}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{4+x^2}}$$

Ans.

EXERCISE 2.7

Q.1: Find y_2 if

$$\text{(i)} \quad y = 2x^5 - 3x^4 + 4x^3 + x - 2 \quad \text{(ii)} \quad y = (2x + 5)^{\frac{3}{2}}$$

$$\text{(iii)} \quad y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

Solution:

$$\text{(i)} \quad y = 2x^5 - 3x^4 + 4x^3 + x - 2$$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

$$\boxed{y_2 = 40x^3 - 36x^2 + 24x} \quad \text{Ans.}$$

$$\text{(ii)} \quad y = (2x + 5)^{\frac{3}{2}}$$

$$y_1 = \frac{3}{2} (2x + 5)^{\frac{3}{2} - 1} \cdot \frac{d}{dx} (2x + 5)$$

$$y_1 = \frac{3}{2} (2x + 5)^{\frac{1}{2} - 1} \cdot 2$$

$$y_1 = 3(2x + 5)^{\frac{1}{2}}$$

$$y_2 = \frac{3}{2} (2x + 5)^{\frac{1}{2} - 1} \cdot \frac{d}{dx} (2x + 5)$$

$$y_2 = \frac{3}{2} (2x + 5)^{\frac{-1}{2}} \cdot 2$$

$$\boxed{y_2 = \frac{3}{\sqrt{2x + 5}}} \quad \text{Ans.}$$

$$\text{(iii)} \quad y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$y = x^{\frac{1}{2}} + x^{\frac{-1}{2}}$$

Diff. w.r.t. 'x'

$$y_1 = \frac{1}{2} x^{\frac{-1}{2}} + \left(\frac{-1}{2} \right) x^{\frac{-3}{2}}$$

$$y_1 = \frac{1}{2} x^{\frac{-1}{2}} - \frac{1}{2} x^{\frac{-3}{2}}$$

$$y_2 = \frac{1}{2} \left(\frac{-1}{2} \right) x^{\frac{-3}{2}} - \left(\frac{-1}{2} \right) \left(\frac{-3}{2} \right) x^{\frac{-5}{2}}$$



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$$y_2 = \frac{-\frac{1}{3}}{4x^2} + \frac{\frac{3}{5}}{4x^2}$$

$$y_2 = \frac{-x + 3}{4x^2}$$

Ans.

Q.2: Find y_2 if

(i) $y = x^2 \cdot e^{-x}$ (L.B 2011) (L.B 2008)

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$ (L.B 2009)

Solution:

(i) $y = x^2 \cdot e^{-x}$

$$y_1 = x^2 \frac{d}{dx}(e^{-x}) + e^{-x} \frac{d}{dx}(x^2)$$

$$y_1 = x^2 \cdot e^{-x} (-1) + e^{-x} 2x$$

$$y_1 = -x^2 e^{-x} + 2x e^{-x}$$

$$y_1 = e^{-x} (2x - x^2)$$

$$y_2 = e^{-x} \frac{d}{dx}(2x - x^2) + (2x - x^2) \cdot \frac{d}{dx}(e^{-x})$$

$$y_2 = e^{-x} (2 - 2x) + (2x - x^2) \cdot e^{-x} \cdot (-1)$$

$$y_2 = e^{-x} (2 - 2x - 2x + x^2)$$

$$y_2 = e^{-x} (x^2 - 4x + 2)$$

Ans.

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

$$y = \ln(2x+3) - \ln(3x+2)$$

$$y_1 = \frac{1}{2x+3} \cdot 2 - \frac{1}{3x+2} \cdot 3$$

$$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

$$y_2 = 2(-1(2x+3)^{-2})(2) - 3(-1(3x+2)^{-2}) \cdot 3$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$y_2 = \frac{-4(3x+2)^2 + 9(2x+3)^2}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{-4(9x^2 + 4 + 12x)^2 + 9(4x^2 + 9 + 12x)}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{-36x^2 - 16 - 48x + 36x^2 + 81 + 108x}{(2x+3)^2(3x+2)^2}$$

$$\boxed{y_2 = \frac{60x + 65}{(2x+3)^2(3x+2)^2}} \quad \text{Ans.}$$

Q.3: Find y_2 if

- (i) $x^2 + y^2 = a^2$
- (ii) $x^3 - y^3 = a^3$
- (iii) $x = a \cos \theta, y = a \sin \theta$ (G.B 2006)
- (iv) $x = at^2, y = bt^4$
- (v) $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution:

$$(i) \quad x^2 + y^2 = a^2$$

$$2x + 2yy_1 = 0$$

$$2yy_1 = -2x$$

$$y_1 = \frac{-2x}{2y}$$

$$y_1 = \frac{-x}{y}$$

$$y_2 = -\frac{y \cdot 1 - x \cdot y_1}{y^2}$$

$$y_2 = -\frac{y - x \left(\frac{-x}{y} \right)}{y^2}$$

$$y_2 = -\frac{y + \frac{x^2}{y}}{y^2}$$

$$= -\frac{y^2 + x^2}{y^3}$$

$$y_2 = -\frac{y^2 + x^2}{y^3}$$

$$\boxed{y_2 = \frac{-a^2}{y^3}} \quad \because x^2 + y = a^2 \quad \text{Ans.}$$

$$(ii) \quad x^3 - y^3 = a^3$$

$$3x^2 - 3y^2 y_1 = 0$$

$$-3y^2 y_1 = -3x^2$$

$$y_1 = \frac{-3x^2}{-3y^2}$$

$$y_1 = \frac{x^2}{y^2}$$

$$y_2 = \frac{y^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(y^2)}{(y^2)^2}$$

$$y_2 = \frac{y^2 \cdot 2x - x^2 \cdot 2yy_1}{y^4}$$

$$y_2 = \frac{2xy^2 - 2x^2y \left(\frac{x^2}{y^2} \right)}{y^4}$$

$$y_2 = \frac{2xy^2 - \frac{2x^4}{y}}{y^4}$$

$$y_2 = \frac{2xy^3 - 2x^4}{y^4}$$

$$y_2 = \frac{-2x(x^3 - y^3)}{y^5}$$

$$y_2 = \boxed{\frac{-2x a^3}{y^5}}$$

Ans.

 $\therefore x^3 - y^3 = a^3$

$$(iii) \quad x = a \cos \theta, \quad y = a \sin \theta$$

$$x = a \cos \theta$$

Diff. w.r.t. ' θ '

$$\frac{dx}{d\theta} = -a \sin \theta$$

$$y = a \sin \theta$$

Diff. w.r.t. ' θ '

$$\frac{dy}{d\theta} = a \cos \theta$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$y_1 = a \cos \theta \cdot \frac{-1}{a \sin \theta}$$

$$y_1 = -\cot \theta$$

$$y_2 = -(-\operatorname{cosec}^2 \theta) \cdot \frac{-1}{a \sin \theta}$$

$$\boxed{y_2 = \frac{-1}{a \sin^3 \theta}} \quad \text{Ans.}$$

(iv) $x = at^2, y = bt^4$

$$x = at^2$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = 2at$$

$$y = bt^4$$

Diff. w.r.t. 't'

$$\frac{dy}{dt} = 4bt^3$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$y_1 = 4bt^3 \cdot \frac{1}{2at}$$

$$y_1 = \frac{2b}{a} t^2$$

$$y_2 = \frac{x}{a} (2t) \frac{dt}{dx} \Rightarrow y_2 = \frac{4bt}{a} \cdot \frac{1}{2at}$$

$$\boxed{y_2 = \frac{2b}{a^2}}$$

Ans.

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

$$2x^2 + 2yy_1 + 2g + 2fy_1 = 0$$

$$2(y + f)y_1 = -2x - 2g$$

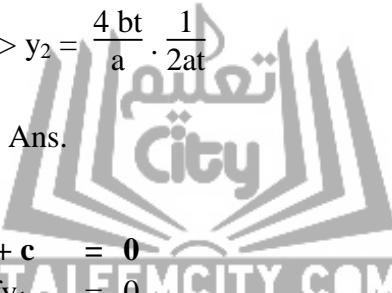
$$y_1 = \frac{-2(x + g)}{2(y + f)}$$

$$y_1 = -\frac{x + g}{y + f}$$

$$y_2 = -\frac{(y + f) \frac{d}{dx}(x + g) - (x + g) \frac{d}{dx}(y + f)}{(y + f)^2}$$

$$y_2 = -\frac{(y + f) - (x + g) \cdot y_1}{(y + f)^2}$$

$$y_2 = -\frac{(y + f) - (x + g) \cdot \left(\frac{-(x + g)}{y + f}\right)}{(y + f)^2}$$



$$\begin{aligned}
 y_2 &= -\frac{(y+f)^2 + (x+g)^2}{y+f} \\
 y_2 &= -\frac{y^2 + f^2 + 2fy + x^2 + g^2 + 2gx}{(y+f)^3} \\
 y_2 &= -\frac{-c + f^2 + g^2}{(y+f)^3} \quad \left[\because x^2 + y^2 + 2gx + 2fy + c = 0 \right. \\
 y_2 &= \boxed{\frac{c - f^2 - g^2}{(y+f)^3}} \quad \text{Ans.}
 \end{aligned}$$

Q.4: Find y_4 if

(i) $y = \sin 3x$ (ii) $y = \cos^3 x$ (iii) $y = \ln(x^2 - 9)$

Solution:

(i) $y = \sin 3x$

$$y_1 = \cos 3x \cdot 3$$

$$y_1 = 3 \cos 3x$$

$$y_2 = 3(-\sin 3x) \cdot 3$$

$$y_2 = -9 \sin 3x$$

$$y_3 = -9 \cos 3x \cdot 3$$

$$y_3 = -27 \cos 3x$$

$$y_4 = -27(-\sin 3x)(3)$$

$$\boxed{y_4 = 81 \sin 3x} \quad \text{Ans.}$$



(ii) $y = \cos^3 x \dots \dots \dots \quad (1) \quad (L.B 2008)$

$$\cos 3x = \cos(2x + x)$$

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x$$

$$\cos 3x = (2\cos^2 x - 1) \cos x - 2 \sin x \cos x \cdot \sin x$$

$$\cos 3x = 2\cos^3 x - \cos x - 2 \cos x \sin^2 x$$

$$\cos 3x = 2\cos^3 x - \cos x - 2 \cos x (1 - \cos^2 x)$$

$$\cos 3x = 2\cos^3 x - \cos x - 2 \cos x + 2\cos^3 x$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$

$$4 \cos^3 x = \cos 3x + 3 \cos x$$

$$\cos^3 x = \frac{1}{4} (\cos 3x + 3 \cos x)$$

\therefore From equation (1)

$$y = \frac{1}{4}(\cos 3x + 3 \cos x)$$

$$y_1 = \frac{1}{4}[-\sin 3x \cdot 3 + 3(-\sin x)]$$

$$y_1 = \frac{1}{4}[-3 \sin 3x - 3 \sin x]$$

$$y_2 = \frac{1}{4}[-3 \cos 3x \cdot 3 - 3 \cos x]$$

$$y_2 = \frac{1}{4}[-9 \cos 3x - 3 \cos x]$$

$$y_3 = \frac{1}{4}[-9(-\sin 3x) \cdot 3 - 3(-\sin x)]$$

$$y_3 = \frac{1}{4}[27 \sin 3x + 3 \sin x]$$

$$y_4 = \frac{1}{4}[27 \cos 3x \cdot 3 + 3 \cos x]$$

$$y_4 = \frac{1}{4}[81(4 \cos^3 x - 3 \cos x) + 3 \cos x]$$

$$y_4 = \frac{1}{4}[324 \cos^3 x - 243 x + 3 \cos x]$$

$$y_4 = \frac{1}{4}[324 \cos^3 x - 240 \cos x]$$

$$y_4 = \frac{4}{4}[81 \cos^3 x - 60 \cos x]$$

$$\boxed{y_4 = -60 \cos x + 81 \cos^3 x} \quad \text{Ans.}$$

(iii) $y = \ell n(x^2 - 9)$

$$y = \ell n(x+3)(x-3)$$

$$y = \ell n(x+3) + \ell n(x-3)$$

$$y_1 = \frac{1}{(x+3)} + \frac{1}{(x-3)}$$

$$y_1 = (x+3)^{-1} + (x-3)^{-1}$$

$$y_2 = -(x+3)^{-2} - (x-3)^{-2}$$

$$y_3 = 2(x+3)^{-3} + 2(x-3)^{-3}$$

$$y_4 = -6(x+3)^{-4} - 6(x-3)^{-4}$$

$$y_4 = \frac{-6}{(x+3)^4} - \frac{6}{(x-3)^4}$$

$y_4 = -6 \left[\frac{1}{(x+3)^4} + \frac{1}{(x-3)^4} \right]$

Ans.

Q.5: If $x = \sin \theta$, $y = \sin m\theta$, Show that $(1-x^2)y_2 - xy_1 + m^2y = 0$
(G.B 2006)

Solution:

$$\begin{aligned} x &= \sin \theta, \quad y = \sin m\theta \\ \theta &= \sin^{-1} x \\ y &= \sin m\theta \\ y &= \sin m(\sin^{-1} x) \\ \sin^{-1} y &= m \sin^{-1} x \end{aligned}$$

$$\frac{1}{\sqrt{1-y^2}} \cdot y_1 = \frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} y_1 = m \sqrt{1-y^2}$$

Squaring on both sides

$$(1-x^2)y_1^2 = m^2(1-y^2)$$

$$(1-x^2) \cdot 2y_1 y_2 + y_1^2 \cdot (-2x) = m^2(-2yy_1)$$

$$2y_1 [(1-x^2) \cdot y_2 - xy_1] = -2m^2yy_1$$

$$(1-x^2) \cdot y_2 - xy_1 = \frac{-2m^2yy_1}{2y_1}$$

$$(1-x^2) \cdot y_2 - xy_1 = -m^2y$$

$(1-x^2)y_2 - xy_1 + m^2y = 0$

Hence proved.

Q.6: If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$ (L.B 2009)

Solution:

$$y = e^x \sin x$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = e^x \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} (e^x)$$

$$\begin{aligned} \frac{dy}{dx} &= e^x \cos x + \sin x e^x \\ &= e^x (\cos x + \sin x) \end{aligned}$$

Diff. again w.r.t. 'x'

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx} (\cos x + \sin x) + (\cos x + \sin x) \frac{d}{dx} (e^x)$$

$$\frac{d^2y}{dx^2} = e^x (-\sin x + \cos x) + (\cos x + \sin x) e^x$$

$$\frac{d^2y}{dx^2} = e^x (-\sin x + \cos x + \cos x + \sin x)$$

$$\frac{d^2y}{dx^2} = 2e^x \cos x$$

Taking

$$\begin{aligned} \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y &= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x \\ &= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x \end{aligned}$$

$$\boxed{\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0}$$

Hence proved.

Q.7: If $y = e^{ax} \sin bx$, show that $\frac{d^2y}{dx^2} - 2a \frac{d^2y}{dx^2} + (a^2 + b^2)y = 0$

Solution:

$$y = e^{ax} \sin bx$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = e^{ax} \frac{d}{dx} (\sin bx) + \sin bx \frac{d}{dx} (e^{ax})$$

$$\frac{dy}{dx} = e^{ax} \cos bx \cdot b + \sin bx \cdot e^{ax} \cdot a$$

$$\frac{dy}{dx} = e^{ax} (b \cos bx + a \sin bx)$$

Diff. again w.r.t. 'x'

$$\frac{d^2y}{dx^2} = e^{ax} (-b \sin bx \cdot b + a \cos bx \cdot b) + (b \cos bx + a \sin bx) e^{ax} \cdot a$$

$$\frac{d^2y}{dx^2} = e^{ax} (-b^2 \sin bx + ab \cos bx + ab \cos bx + a^2 \sin bx)$$

$$\frac{d^2y}{dx^2} = e^{ax} (-b^2 \sin bx + 2ab \cos bx + a^2 \sin bx)$$

Taking

$$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = e^{ax} (-b^2 \sin bx + 2ab \cos bx + a^2 \sin bx) - 2ae^{ax}$$

$$\begin{aligned}
 & (b \cos bx + a \sin bx) + (a^2 + b^2) e^{ax} \sin bx \\
 \frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y &= -e^{ax} \sin bx + 2e^{ax} ab \cos bx + e^{ax} a^2 \sin bx - 2abe^{ax} \\
 &\quad \cos bx - 2a^2 e^{ax} \sin bx + a^2 e^{ax} \sin bx + b^2 e^{ax} \sin bx \\
 \boxed{\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0} & \text{ Hence proved.}
 \end{aligned}$$

Q.8: If $y = (\cos^{-1} x)^2$, prove that $(1 - x^2) y_2 - xy_1 - 2 = 0$ (G.B 2007)

Solution:

$$y = (\cos^{-1} x)^2$$

$$y_1 = 2(\cos^{-1} x) \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} y_1 = -2 (\cos^{-1} x)$$

Squaring on both sides

$$(1 - x^2) y_1^2 = 4 (\cos^{-1} x)^2 \quad \because y = (\cos^{-1} x)^2$$

$$(1 - x^2) y_1^2 = 4y$$

$$(1 - x^2) \cdot 2y_1 y_2 + y_1^2 \cdot (-2x) = 4y_1$$

$$2y_1 [(1 - x^2) y_2 - xy_1] = 4y_1$$

$$(1 - x^2) y_2 - xy_1 = \frac{4y_1}{2y_1}$$

$$(1 - x^2) y_2 - xy_1 = 2$$

$$\boxed{(1 - x^2) y_2 - xy_1 - 2 = 0} \quad \text{Hence proved.}$$

Q.9: If $y = a \cos(\ell \ln x) + b \sin(\ell \ln x)$ Prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Solution:

$$y = a \cos(\ell \ln x) + b \sin(\ell \ln x)$$

$$\frac{dy}{dx} = -a \sin(\ell \ln x) \cdot \frac{1}{x} + b \cos(\ell \ln x) \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x} [-a \sin(\ell \ln x) + b \cos(\ell \ln x)]$$

$$\frac{d^2y}{dx^2} = \frac{1}{x} [-a \cos(\ell \ln x) \cdot \frac{1}{x} - b \sin(\ell \ln x) \cdot \frac{1}{x}]$$

$$+ [(-a \sin(\ell \ln x) + b \cos(\ell \ln x)) \left(\frac{-1}{x^2} \right)]$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) - \frac{1}{x^2} [-a \sin(\ln x) + b \cos(\ln x)]]$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) - b \cos(\ln x)]$$

Taking

$$\begin{aligned} x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y &= x^2 \cdot \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) \\ &\quad - b \cos(\ln x)] + x \cdot \frac{1}{x} [-a \sin(\ln x) + b \cos(\ln x)] + a \cos(\ln x) + b \sin(\ln x) \\ &= -a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) - b \cos(\ln x) - a \sin(\ln x) \\ &\quad + b \cos(\ln x) + a \cos(\ln x) + b \sin(\ln x) \end{aligned}$$

$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Hence proved.

EXERCISE 2 . 8

Q.1 Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (\text{L.B 2005})$

(ii) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$

(iii) $\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16} + \dots$

(iv) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad (\text{L.B 20011})$

(v) $e^{2x} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \dots$

Solution:

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Let

$$f(x) = \ln(1+x)$$

$$f(0) = \ln(1+0) = \ln 1 = 0$$