

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) - \frac{1}{x^2} [-a \sin(\ln x) + b \cos(\ln x)]]$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) - b \cos(\ln x)]$$

Taking

$$\begin{aligned} x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y &= x^2 \cdot \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) \\ &\quad - b \cos(\ln x)] + x \cdot \frac{1}{x} [-a \sin(\ln x) + b \cos(\ln x)] + a \cos(\ln x) + b \sin(\ln x) \\ &= -a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) - b \cos(\ln x) - a \sin(\ln x) \\ &\quad + b \cos(\ln x) + a \cos(\ln x) + b \sin(\ln x) \end{aligned}$$

$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Hence proved.

EXERCISE 2 . 8

Q.1 Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (\text{L.B 2005})$

(ii) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$

(iii) $\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16} + \dots$

(iv) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad (\text{L.B 20011})$

(v) $e^{2x} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \dots$

Solution:

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Let

$$f(x) = \ln(1+x)$$

$$f(0) = \ln(1+0) = \ln 1 = 0$$

$f^I(x) = \frac{1}{1+x}$	$f^I(0) = \frac{1}{1+0} = 1$
$f^I(x) = (1+x)^{-1}$	$f^{II}(0) = -(1+0)^{-2} = -1$
$f^{II}(x) = -(1+x)^{-2}$	$f^{III}(0) = 2(1+0)^{-3} = 2$
$f^{III}(x) = 2(1+x)^{-3}$	$f^{IV}(0) = -6(1+0)^{-4} = -6$
$f^{IV}(x) = -6(1+x)^{-4}$	

The Maclaurin series expansion is

$$\begin{aligned} f(x) &= f(0) + x f^I(0) + \frac{x^2}{2!} f^{II}(0) + \frac{x^3}{3!} f^{III}(0) + \frac{x^4}{4!} f^{IV}(0) + \dots \\ &= 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \dots \end{aligned}$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Hence proved.

(ii) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$

Let $f(x) = \cos x$

$$f(0) = \cos 0 = 1$$

$$f^I(x) = -\sin x$$

$$f^{II}(x) = -\cos x$$

$$f^{III}(x) = \sin x$$

$$f^{IV}(x) = \cos x$$

$$f^V(x) = -\sin x$$

$$f^{VI}(x) = -\cos x$$

$$f^I(0) = -\sin 0 = 0$$

$$f^{II}(0) = -\cos 0 = -1$$

$$f^{III}(0) = \sin 0 = 0$$

$$f^{IV}(0) = \cos 0 = 1$$

$$f^V(0) = -\sin 0 = 0$$

$$f^{VI}(0) = -\cos 0 = -1$$

The Maclaurin series expansion is

$$\begin{aligned} f(x) &= f(0) + x f^I(0) + \frac{x^2}{2!} f^{II}(0) + \frac{x^3}{3!} f^{III}(0) + \frac{x^4}{4!} f^{IV}(0) + \frac{x^5}{5!} f^V(0) + \frac{x^6}{6!} f^{VI}(0) + \dots \\ &= 1 + x(0) + \frac{x^2}{2}(-1) + \frac{x^3}{3}(0) + \frac{x^4}{4}(1) + \frac{x^5}{5}(0) + \frac{x^6}{6}(-1) + \dots \end{aligned}$$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$

Hence proved.

(iii) $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^6}{16} + \dots$

Let $f(x) = \sqrt{1+x}$

$$\begin{array}{l}
 f(0) = \sqrt{1+0} \\
 = \sqrt{1} = 1 \\
 f'(x) = \frac{1}{2} (x+1)^{\frac{-1}{2}} \\
 f''(x) = -\frac{1}{4} (1+x)^{\frac{-3}{2}} \\
 f'''(x) = \frac{3}{8} (1+x)^{\frac{-5}{2}}
 \end{array}
 \quad \left| \quad \begin{array}{l}
 f'(0) = \frac{1}{2} (1+0)^{\frac{-1}{2}} = \frac{1}{2} \\
 f''(0) = -\frac{1}{4} (1+0)^{\frac{-3}{2}} = -\frac{1}{4} \\
 f'''(0) = \frac{3}{8} (1+0)^{\frac{-5}{2}} = \frac{3}{8}
 \end{array} \right.$$

The Maclaurin series expansion is

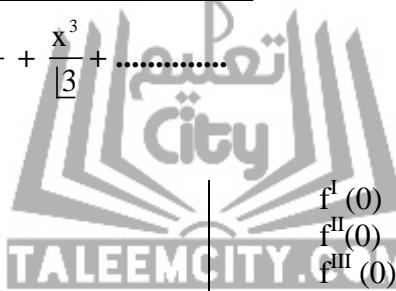
$$\begin{aligned}
 f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \\
 &= 1 + x\left(\frac{1}{2}\right) + \frac{x^2}{2}\left(-\frac{1}{4}\right) + \frac{x^3}{6}\left(\frac{3}{8}\right) + \dots
 \end{aligned}$$

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots}$$

Hence proved.

$$(iv) \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$\begin{aligned}
 \text{Let } f(x) &= e^x \\
 f(0) &= e^0 = 1 \\
 f'(x) &= e^x \\
 f''(x) &= e^x \\
 f'''(x) &= e^x
 \end{aligned}$$



$$\begin{aligned}
 f'(0) &= e^0 = 1 \\
 f''(0) &= e^0 = 1 \\
 f'''(0) &= e^0 = 1
 \end{aligned}$$

The Maclaurin series expansion is

$$\begin{aligned}
 f(x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3}f'''(0) + \dots \\
 &= 1 + x(1) + \frac{x^2}{2}(1) + \frac{x^3}{3}(1) + \dots
 \end{aligned}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots}$$

Hence proved.

$$(v) \quad e^{2x} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \dots$$

Let

$$\begin{aligned}
 f(x) &= e^{2x} \\
 f(0) &= e^{2(0)} = e^0 = 1
 \end{aligned}$$

$f'(x) = 2e^{2x}$	$f'(0) = 2e^{2(0)} = 2e^0 = 2$
$f''(x) = 4e^{2x}$	$f''(0) = 4e^{2(0)} = 4e^0 = 4$
$f'''(x) = 8e^{2x}$	$f'''(0) = 8e^{2(0)} = 8e^0 = 8$

The Maclaurin series expansion is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3} f'''(0) + \dots$$

$$e^{2x} = 1 + x(2) + \frac{x^2}{2}(4) + \frac{x^3}{3}(8) + \dots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \dots$$

Hence proved.

Q.2: Show that

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2} \cos x + \frac{h^3}{3} \sin x + \dots \text{ and evaluate } \cos 61^\circ.$$

Solution:

Let $f(x+h) = \cos(x+h)$

then

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$



The Taylor series expansion is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3} f'''(x) + \dots$$

$$\cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2}(-\cos x) + \frac{h^3}{3}\sin x + \dots$$

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2} \cos x + \frac{h^3}{3} \sin x + \dots$$

Put $x = 60^\circ$, $h = 1^\circ = \frac{\pi}{180}$ rad = 0.01745 rad

$$\begin{aligned} \cos(60^\circ + 1^\circ) &= \cos 60^\circ - (0.01745) \sin 60^\circ \\ &\quad - \frac{(0.01745)^2}{2} \cos 60^\circ + \frac{(0.01745)^3}{6} \sin 60^\circ + \dots \end{aligned}$$

$$\cos 61^\circ \approx 0.5 - 0.0151 - 0.000076 + 0.00000076 + \dots$$

$\cos 61^\circ \approx 0.4848$	Ans
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Q.3: Show that

$$2^{x+h} = 2^x \left\{ 1 + (\ln 2)h + \frac{(\ln 2)^2}{2} h^2 + \frac{(\ln 2)^3}{3} h^3 + \dots \right\}$$

Solution:

Let

$$f(x+h) = 2^{x+h}$$

then

$$f(x) = 2^x$$

$$f'(x) = (\ln 2) 2^x$$

$$f''(x) = (\ln 2)^2 2^x$$

$$f'''(x) = (\ln 2)^3 2^x$$

The Taylor series expansion is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3} f'''(x) + \dots$$

$$2^{x+h} = 2^x + h (\ln 2) 2^x + \frac{h^2}{2} (\ln 2)^2 2^x + \frac{h^3}{3} (\ln 2)^3 2^x + \dots$$

$$2^{x+h} = 2^x \left\{ 1 + (\ln 2)h + \frac{(\ln 2)^2}{2} h^2 + \frac{(\ln 2)^3}{3} h^3 + \dots \right\}$$

Hence proved.