EXERCISE 2.9

Q.1: Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case.

(i) $f(x) = \sin x \quad ; \quad x \in (-\pi, \pi)$

(ii) $f(x) = \cos x \quad ; \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ***(L.B 2005)***

(iii) $f(x) = 4 - x^2 \quad ; \quad x \in (-2, 2)$

(iv) $f(x) = x^2 + 3x + 2 \quad ; \quad x \in (-4, 1)$

Solution:

$f(x) = \sin x \quad ; \quad x \in (-\pi, \pi)$

$f'(x) = \cos x$

Put

$f'(x) = 0$

=> $\cos x = 0$

=> $x = \frac{-\pi}{2}, \frac{\pi}{2}$

So the sub-intervals are

$\left(-\pi, \frac{-\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right)$

For $\left(-\pi, \frac{-\pi}{2}\right)$

$f'(x) = \cos x < 0$ in $\left(-\pi, \frac{-\pi}{2}\right)$

So f(x) is decreasing in $\left(-\pi, \frac{-\pi}{2}\right)$

For $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$f'(x) = \cos x > 0$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

So f(x) is increasing in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

For $\left(\frac{\pi}{2}, \pi\right)$

$f'(x) = \cos x < 0 \quad \text{in } (\frac{\pi}{2}, \pi)$

So $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(ii) $f(x) = \cos x \quad ; \quad x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$f'(x) = -\sin x$

Put

$f'(x) = 0$

$\sin x = 0$

$x = 0$

So the sub-intervals are $\left(\frac{-\pi}{2}, 0\right), \left(0, \frac{\pi}{2}\right)$

For $\left(\frac{-\pi}{2}, 0\right)$

$f'(x) = -\sin x > 0 \quad \text{in } \left(\frac{-\pi}{2}, 0\right)$

So f is increasing in $\left(\frac{-\pi}{2}, 0\right)$

For $\left(0, \frac{\pi}{2}\right)$

$f'(x) = -\sin x < 0 \quad \text{in } \left(0, \frac{\pi}{2}\right)$

So f is decreasing in $\left(0, \frac{\pi}{2}\right)$

(iii) $f(x) = 4 - x^2 \quad ; \quad x \in (-2, 2)$ ***(L.B 2008)***

$f'(x) = -2x$

Put

$f'(x) = 0$

$-2x = 0$

$x = 0$

So the sub-intervals are $(-2, 0)$ and $(0, 2)$

For $(-2, 0)$

$f'(x) = -2x > 0 \quad \text{in} \quad (-2, 0)$

So $f(x)$ is increasing in $(-2, 0)$

For $(0, 2)$

$f'(x) = -2x < 0 \quad \text{in} \quad (0, 2)$

So f(x) is decreasing in (0, 2)

(iv) $f(x) = x^2 + 3x + 2 \quad ; \; x \in (-4, 1)$ ***(L.B 2007) (G.B 2008)***

$f'(x) = 2x + 3$

Put $f'(x) = 0$

$2x + 3 = 0$

$2x = -3$

$x = \dfrac{-3}{2}$

So the sub-intervals are $\left(-4 \;,\; \dfrac{-3}{2}\right) \;,\; \left(\dfrac{-3}{2} \;,\; 1\right)$

For $\left(-4 \;,\; \dfrac{-3}{2}\right)$

$f'(x) = 2x + 3 < 0 \;$ in $\left(-4 , \dfrac{-3}{2}\right)$

So f (x) is decreasing in $\left(-4 \;,\; \dfrac{-3}{2}\right)$

For $\left(\dfrac{-3}{2} \;,\; 1\right)$

$f'(x) = 2x + 3 > 0 \;$ in $\left(\dfrac{-3}{2} \;,\; 1\right)$

So f (x) is increasing in $\left(\dfrac{-3}{2} \;,\; 1\right)$

Q.2: Find the extreme values of the following functions defined as

(i) $f(x) = 1 - x^3$ **(ii)** $f(x) = x^2 - x - 2$

(iii) $f(x) = 5x^2 - 6x + 2$ **(iv)** $f(x) = 3x^2$

(v) $f(x) = 3x^2 - 4x + 5$ **(vi)** $f(x) = 2x^3 - 2x^2 - 36x + 3$

(vii) $f(x) = x^4 - 4x^2$ **(viii)** $f(x) = (x-2)^2 (x-1)$

(ix) $f(x) = 5 + 3x - x^3$ ***(L.B 2011)***

Solution:

(i) $f(x) = 1 - x^3$

$f'(x) = -3x^2$

$f''(x) = -6x$

For stationary points

Put $f'(x) = 0$

$-3x^2 = 0$

$$x^2 = 0$$
$$x = 0$$

The second derivative does not help in determining the extreme values.

Before $x = 0$, $f'(x) < 0$

After $x = 0$, $f'(x) < 0$

$\therefore$ $x = 0$ has a point of inflection.

Put

$x = 0$ in

$$f(x) = 1 - x^3$$
$$f(0) = 1 - (0)^3 = 1$$

$\therefore$ Point of inflection is $(0, 1)$

(ii) $\mathbf{f(x) = x^2 - x - 2}$

$$f'(x) = 2x - 1$$
$$f''(x) = 2$$

For stationery points

Put $f'(x) = 0$

$$2x - 1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2}$$

Put $x = \frac{1}{2}$ in $f''(x)$, we get

$$f''\left(\frac{1}{2}\right) = 2 > 0$$

$\therefore$ f has relative minima at $x = \frac{1}{2}$

Put $x = \frac{1}{2}$ in

$$f(x) = x^2 - x - 2$$
$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2$$
$$= \frac{1}{4} - \frac{1}{2} - 2$$
$$= \frac{1-2-8}{4} = \frac{-9}{4} \qquad \text{Ans.}$$

(iii) ***(L.B 2009 (s)) (L.B 2009)***

$$f(x) = 5x^2 - 6x + 2$$
$$f'(x) = 10x - 6$$
$$f''(x) = 10$$

For stationary points

Put

$$f'(x) = 0$$
$$10x - 6 = 0$$
$$10x = 6$$
$$x = \frac{6}{10}$$
$$= \frac{3}{5}$$

Put

$$x = \frac{3}{5} \text{ in } f''(x), \quad \text{we get}$$
$$f''\left(\frac{3}{5}\right) = 10 > 0$$

$\therefore$ f has relative minima at $x = \frac{3}{5}$

Put $x = \frac{3}{5}$ in

$$f(x) = 5x^2 - 6x + 2$$
$$f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2$$
$$= 5\left(\frac{9}{25}\right) - \frac{18}{5} + 2$$
$$= \frac{9}{5} - \frac{18}{5} + 2$$
$$= \frac{9 - 18 + 10}{5}$$
$$\boxed{f\left(\frac{3}{5}\right) = \frac{1}{5}} \quad \text{Ans}$$

(iv) $\mathbf{f(x) = 3x^2}$

$f'(x) = 6x$

$f''(x) = 6$

For stationary points

Put $f'(x) = 0$

$6x = 0$

$x = 0$

Put $x = 0$ in $f''(x)$, we get

$f''(0) = 6 > 0$

$\therefore$ f has relative minima at $x = 0$

Put $x = 0$ in

$f(x) = 3x^2$

$f(0) = 3(0)^2$

$\boxed{f(0) = 0}$ Ans

(v) ***(G.B 2008)***

$f(x) = 3x^2 - 4x + 5$

$f'(x) = 6x - 4$

$f''(x) = 6$

For stationary points

Put $f'(x) = 0$

$6x - 4 = 0$

$6x = 4$

$x = \frac{4}{6}$

$= \frac{2}{3}$

Put $x = \frac{2}{3}$ in $f''(x)$, we get

$f''\left(\frac{2}{3}\right) = 6 > 0$

$\therefore$ f has relative minima at $x = \frac{2}{3}$

Put $x = \frac{2}{3}$ in

$f(x) = 3x^2 - 4x + 5$

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5$$

$$= 3\left(\frac{4}{9}\right) - \frac{8}{3} + 5$$

$$= \frac{4}{3} - \frac{8}{3} + 5$$

$$= \frac{4 - 8 + 15}{3}$$

$$\boxed{f\left(\frac{2}{3}\right) = \frac{11}{3}} \text{ Ans}$$

(vi) ***(G.B 2005)***

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$
$$f'(x) = 6x^2 - 4x - 36$$
$$f''(x) = 12x - 4$$

For Stationary Points

Put

$$f'(x) = 0$$
$$6x^2 - 4x - 36 = 0$$
$$2(3x^2 - 2x - 18) = 0$$
$$3x^2 - 2x - 18 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 3$, $b = -2$, $c = -18$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-18)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 216}}{6}$$

$$x = \frac{2 \pm \sqrt{220}}{6}$$

$$x = \frac{2 \pm \sqrt{4 \times 55}}{6}$$

$$x = \frac{2 \pm 2\sqrt{55}}{6}$$

$$x = \frac{2(1 \pm \sqrt{55})}{6}$$

$$x = \frac{1 \pm \sqrt{55}}{3}$$

Put $x = \dfrac{1 + \sqrt{55}}{3}$ in $f''(x)$, we get

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 12\left(\frac{1+\sqrt{55}}{3}\right) - 4$$

$$= 4(1 + \sqrt{55}) - 4$$

$$= 4 + 4\sqrt{55} - 4$$

$$= 4\sqrt{55} > 0$$

$\therefore$ f has relative minima at $x = \dfrac{1+\sqrt{55}}{3}$

Put $x = \dfrac{1+\sqrt{55}}{3}$ in

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$$= 2\ \frac{[1 + 55\sqrt{55} + 3\sqrt{55} + 3(55)]}{27} - 2\left(\frac{1 + 55 + 2\sqrt{55}}{9}\right) - 12(1 + \sqrt{55}) + 3$$

$$= \frac{2(166 + 58\sqrt{55})}{27} - 2\left(\frac{56 + 2\sqrt{55}}{9}\right) - 12 - 12\sqrt{55} + 3$$

$$= \frac{332 + 116\sqrt{55}}{27} - \frac{112 + 4\sqrt{55}}{9} - 12\sqrt{55} - 9$$

$$= \frac{332 + 116\sqrt{55} - 3(112 + 4\sqrt{55}) - 324\sqrt{55} - 243}{27}$$

$$= \frac{89 - 208\sqrt{55} - 336 - 12\sqrt{55}}{27}$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{1}{27}(-247 - 220\sqrt{55})$$

Put $x = \dfrac{1-\sqrt{55}}{3}$ in $f''(x)$, we get

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 12\left(\frac{1-\sqrt{55}}{3}\right) - 4$$

$$= 4(1 - \sqrt{55}) - 4 = 4 - 4\sqrt{55} - 4 = -4\sqrt{55} < 0$$

$\therefore$ f has relative maxima at $x = \dfrac{1-\sqrt{55}}{3}$

Put $x = \dfrac{1-\sqrt{55}}{3}$ in

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$

$$= 2\left(\frac{1-\sqrt{55}}{3}\right)^3 - 2\left(\frac{1-\sqrt{55}}{3}\right)^2 - 36\left(\frac{1-\sqrt{55}}{3}\right) + 3$$

$$= \frac{2\,[1 - 55\sqrt{55} - 3\sqrt{55} + 3\,(55)]}{27} - 2\left(\frac{1 + 55 - 2\sqrt{55}}{9}\right) - 12\,(1 - \sqrt{55}) + 3$$

$$= \frac{2(166 - 58\sqrt{55})}{27} - 2\left(\frac{56 - 2\sqrt{55}}{9}\right) - 12 + 12\sqrt{55} + 3$$

$$= \frac{332 - 116\sqrt{55}}{27} - \frac{112 - 4\sqrt{55}}{9} + 12\sqrt{55} - 9$$

$$= \frac{332 - 116\sqrt{55} - 3(112 - 4\sqrt{55}) + 324\sqrt{55} - 243}{27}$$

$$= \frac{89 + 208\sqrt{55} - 336 + 12\sqrt{55}}{27}$$

$$f\left(\frac{1-\sqrt{55}}{3}\right) = \frac{1}{27}\,(-247 + 220\sqrt{55})$$

(vii) $\mathbf{f(x) = x^4 - 4x^2}$

$f'(x) = 4x^3 - 8x$

$f''(x) = 12x^2 - 8$

For stationary points

Put

$f'(x) = 0$

$4x^3 - 8x = 0$

$4x(x^2 - 2) = 0$

$x(x^2 - 2) = 0$

Either

$x = 0$ or $x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

Put $x = 0$ in $f''(x)$, we get

$f''(0) = 12\,(0)^2 - 8 = -8 < 0$

$\therefore$ f has relative maxima at $x = 0$

Put $x = 0$ in

$$f(x) = x^4 - 4x^2$$

$$f(0) = (0)^4 - 4(0)^2$$

$$\boxed{f(0) = 0}$$

Put

$$x = \sqrt{2} \text{ in } f''(x), \text{ w get}$$

$$f''(\sqrt{2}) = 12(\sqrt{2})^2 - 8$$

$$= 12(2) - 8$$

$$= 24 - 8$$

$$= 16 > 0$$

$\therefore$ f has relative minima at $x = \sqrt{2}$

Put $x = \sqrt{2}$ in

$$f(x) = x^4 - 4x^2$$

$$f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2$$

$$= 4 - 4(2)$$

$$= 4 - 8$$

$$\boxed{f(\sqrt{2}) = -4}$$

Put

$$x = -\sqrt{2} \text{ in } f''(x), \text{ we get}$$

$$f''(-\sqrt{2}) = 12(-\sqrt{2}) - 8$$

$$= 12(2) - 8$$

$$= 24 - 8$$

$$= 16 > 0$$

$\therefore$ f(x) has relative minima at $x = -\sqrt{2}$

Put $x = -\sqrt{2}$ in

$$f(x) = x^4 - 4x^2$$

$$f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2$$

$$= 4 - 4(2)$$

$$= 4 - 8$$

$$\boxed{f(-\sqrt{2}) = -4}$$

(viii)

$$f(x) = (x-2)^2(x-1)$$
$$f(x) = (x^2+4-4x)(x-1)$$
$$f(x) = x^3+4x-4x^2-x^2-4+4x$$
$$f(x) = x^3-5x^2+8x-4$$
$$f'(x) = 3x^2-10x+8$$
$$f''(x) = 6x-10$$

For stationary points

Put

$$f'(x) = 0$$
$$3x^2-10x+8 = 0$$
$$3x^2-6x-4x+8 = 0$$
$$3x(x-2)-4(x-2) = 0$$
$$(x-2)(3x-4) = 0$$

Either

$x-2 = 0$ or $3x-4 = 0$

$x = 2$ $3x = 4$

$$x = \frac{4}{3}$$

Put $x = 2$ in $f''(x)$, we get

$$f''(2) = 6(2)-10$$
$$= 12-10$$
$$= 2 > 0$$

$\therefore$ f has relative minima at $x = 2$

Put $x = 2$ in

$$f(x) = (x-2)^2(x-1)$$
$$f(2) = (2-2)^2(2-1)$$
$$f(2) = (0)^2(1)$$
$$\boxed{f(2) = 0}$$

Put $x = \frac{4}{3}$ in $f''(x)$, we get

$$f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right)-10$$
$$= 8-10 = -2 < 0$$

$\therefore$ f has relative maxima at $x = \frac{4}{3}$

Put $x = \frac{4}{3}$ in

$$f(x) = (x-2)^2(x-1)$$

$$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}-2\right)^2\left(\frac{4}{3}-1\right)$$

$$= \left(\frac{4-6}{3}\right)^2\left(\frac{4-3}{3}\right)$$

$$= \left(\frac{-2}{3}\right)^2\left(\frac{1}{3}\right)$$

$$= \left(\frac{4}{9}\right)\left(\frac{1}{3}\right)$$

$$\boxed{f\left(\frac{4}{3}\right) = \frac{4}{27}}$$

(ix)

$$f(x) = 5 + 3x - x^3$$

$$f'(x) = 3 - 3x^2$$

$$f''(x) = -6x$$

For stationery points

Put $f'(x) = 0$

$$3 - 3x^2 = 0$$

$$-3x^2 = -3$$

$$x^2 = \frac{-3}{-3}$$

$$x^2 = 1$$

$$x = \pm 1$$

Put $x = 1$ in $f''(x)$, we get

$$f''(1) = -6(1) = -6 < 0$$

$\therefore$ f has relative maxima at $x = 1$

Put $x = 1$ in

$$f(x) = 5 + 3x - x^3$$

$$f(1) = 5 + 3(1) - (1)^3$$

$$= 5 + 3 - 1$$

$$f(1) = 7$$

Put $x = -1$ in $f''(x)$, we get

$f''(-1) = -6(-1) = 6 > 0$

$\therefore$ f has relative minima at $x = -1$

Put $x = -1$ in

$f(x) = 5 + 3x - x^3$

$f(-1) = 5 + 3(-1) - (-1)^3$

$= 5 - 3 + 1$

$\boxed{f(-1) = 3}$

Q.3: Find the maximum and minimum values of the function defined by the following equation occuring in the interval [0, 2π].

$f(x) = \sin x + \cos x$

Solution:

$f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

For stationary points

Put $f'(x) = 0$

$\cos x - \sin x = 0$

$\cos x = \sin x$

$\dfrac{\sin x}{\cos x} = 1$

$\tan x = 1$

Since tangent is positive the in 1st and 3rd quadrant with reference angle $\dfrac{\pi}{4}$.

$x = \dfrac{\pi}{4}$, $x = \pi + \dfrac{\pi}{4}$

$x = \dfrac{5\pi}{4}$

Put $x = \dfrac{\pi}{4}$ in $f''(x)$, we get

$$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= \frac{-1-1}{\sqrt{2}}$$

$$= \frac{-2}{\sqrt{2}} < 0$$

$\therefore$ f has relative maxima at $x = \frac{\pi}{4}$

Put $x = \frac{\pi}{4}$ in

$$f(x) = \sin x + \cos x$$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{1+1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{2}}$$

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Put $x = \frac{5\pi}{4}$ in $f''(x)$, we get

$$f''\left(\frac{5\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}$$

$$= \left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{1+1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} > 0$$

$\therefore$ f has relative minima at $x = \frac{5\pi}{4}$

Put $x = \frac{5\pi}{4}$ in

$$f(x) = \sin x + \cos x$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$$

$$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= \frac{-1-1}{\sqrt{2}}$$

$$= \frac{-2}{\sqrt{2}}$$

$$= \frac{-\sqrt{2}\times\sqrt{2}}{\sqrt{2}}$$

$$f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$$

Q.4: Show that $y = \frac{\ell nx}{x}$ has maximum value at $x = e$ ***(L.B 2005)***

Solution:

$$y = \frac{\ell nx}{x}$$

$$\frac{dy}{dx} = \frac{x\,.\,\frac{1}{x} - \ell nx\,.\,1}{x^2}$$

$$\frac{dy}{dx} = \frac{1-\ell nx}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2\,.\,\frac{-1}{x} - (1-\ell nx).2x}{(x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x-2x+2x\,\ell nx}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3x+2x\,\ell nx}{x^4}$$

For stationary points

Put $\frac{dy}{dx} = 0$

$$\frac{1-\ell nx}{x^2} = 0$$

$$1-\ell nx = 0$$

$$1 = \ell\, nx$$

$$\ell nx = \ell ne$$

$$x = e \qquad \because \ \ell\, ne = 1$$

Put $x = e$ in $\frac{d^2y}{d^2x}$, we get

$$\frac{d^2y}{d^2x} = \frac{-3e+2e}{e^4}$$

$$= \frac{-e}{e^4}$$

$$= \frac{-1}{e^3} < 0$$

Shows $y = \frac{\ell nx}{x}$ has maximum value at $x = e$.

Q.5: Show that $y = x^x$ has a minimum value at $x = \frac{1}{e}$ ***(L.B 2006)***

Solution:

$y = x^x$

Taking 'ℓn' on both sides

$$\ell ny = \ell nx^x$$

$$\ell ny = x\,\ell nx$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \ell nx\cdot 1$$

$$\frac{dy}{dx} = y\,[\,1+\ell nx\,]$$

$$\frac{dy}{dx} = x^x\,(\,1+\ell nx\,)$$

$$\frac{d^2y}{dx^2} = x^x\frac{d}{dx}\,(1+\ell nx\,) + (1+\ell nx\,)\frac{d}{dx}\,(x^x)$$

$$\frac{d^2y}{dx^2} = x^x \cdot \frac{1}{x} + (1 + \ell nx) \cdot x^x (1 + \ell nx)$$

$$\frac{d^2y}{dx^2} = x^x \left[\frac{1}{x} + (1 + \ell nx)^2 \right]$$

For stationary points

Put $\dfrac{dy}{dx} = 0$

$$x^x (1 + \ell nx) = 0$$

$$1 + \ell nx = 0 \quad , \quad x^x \neq 0$$

$$\ell ne + \ell nx = 0$$

$$\ell nx = -\ell ne$$

$$\ell nx = \ell ne^{-1}$$

$$x = e^{-1}$$

$$x = \frac{1}{e}$$

Put $x = \dfrac{1}{e}$ in $\dfrac{d^2y}{d^2x}$, we get

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[\frac{1}{\frac{1}{e}} + \left(1 + \ell n\frac{1}{e}\right)^2 \right]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[e + (1 + \ell ne^{-1})^2 \right]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[e + (1 - \ell ne)^2 \right]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[e + (1 - 1)^2 \right]$$

$$\frac{d^2y}{dx^2} = e\left(\frac{1}{e}\right)^{\frac{1}{e}} > 0$$

Shows $y = x^x$ has a minimum value at $x = \dfrac{1}{e}$