

$$\begin{aligned}
 &= - \int (\sec^2 x - 1) dx \\
 &= - \int \sec^2 x dx + \int dx \\
 &= x - \tan x + c \quad \text{Ans.}
 \end{aligned}$$

(xiv)  $\int \tan^2 x dx$  (Guj. Board 2005, 2007) (Lhr. Board 2011)

$$\begin{aligned}
 &= \int (\sec^2 x - 1) dx \\
 &= \int \sec^2 x dx - \int dx \\
 &= \tan x - x + c \quad \text{Ans.}
 \end{aligned}$$

### EXERCISE 3.3

Evaluate the following integrals.

Q.1  $\int \frac{-2x}{\sqrt{4-x^2}} dx$



*Solution:*

$$\begin{aligned}
 &\int \frac{-2x}{\sqrt{4-x^2}} dx \\
 &= \int (4-x^2)^{-1/2} - 2x dx \\
 &= \frac{(4-x^2)^{1/2}}{\frac{1}{2}} + c \\
 &= 2\sqrt{4-x^2} + c \quad \text{Ans.}
 \end{aligned}$$

Q.2  $\int \frac{dx}{x^2 + 4x + 13}$

*Solution:*

$$\int \frac{dx}{x^2 + 4x + 13}$$

$$\begin{aligned}
 &= \int \frac{dx}{x^2 + 4x + 4 - 4 + 13} \\
 &= \int \frac{dx}{(x+2)^2 + 9} \\
 &= \int \frac{dx}{(x+2)^2 + (3)^2} \\
 &= \frac{1}{3} \tan^{-1} \left( \frac{x+2}{3} \right) + c \quad \text{Ans.}
 \end{aligned}$$

**Q.3**  $\int \frac{x^2}{4+x^2} dx$

**Solution:**

$$\begin{aligned}
 &\int \frac{x^2}{4+x^2} dx \\
 &= \int \left( 1 - \frac{4}{4+x^2} \right) dx \\
 &= \int dx - 4 \int \frac{4x}{(2)^2 + x^2} dx \\
 &= x - 4 \int \frac{4x}{4+x^2} dx \\
 &= x - 2 \tan^{-1} \left( \frac{x}{2} \right) + c
 \end{aligned}$$

**Q.4**  $\int \frac{1}{x \ln x} dx$

**Solution:**

$$\begin{aligned}
 &\int \frac{1}{x \ln x} dx \\
 &= \int \frac{1/x}{\ln x} dx \\
 &= \ln(\ln x) + c \quad \text{Ans.}
 \end{aligned}$$

$\therefore \int [f(x)]^{-1} \cdot f'(x) dx = \ln[f(x)] + c$

$f(x) = \ln x$ $f'(x) = \frac{1}{x}$
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**Q.5**  $\int \frac{e^x}{e^x + 3} dx$

**Solution:**

$$\begin{aligned}
 &\int \frac{e^x}{e^x + 3} dx \\
 &= \ln(e^x + 3) + c \quad \text{Ans.}
 \end{aligned}$$

**Q.6**  $\int \frac{x+b}{(x^2+2bx+c)^{1/2}} dx$

**Solution:**

$$\int \frac{x+b}{(x^2+2bx+c)^{1/2}} dx$$

$$\begin{aligned}f(x) &= x^2 + 2bx + c \\f'(x) &= 2x + 2b \\f'(x) &= 2(x + b)\end{aligned}$$

$$= \frac{1}{2} \int (x^2 + 2bx + c)^{-1/2} \cdot 2(x + b) dx$$

$$= \frac{1}{2} \int (x^2 + 2bx + c)^{-1/2} (2x + 2b) dx$$

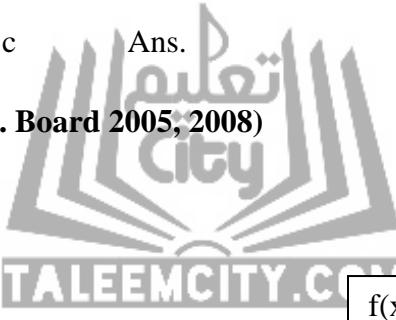
$$= \frac{1}{2} \frac{(x^2 + 2bx + c)^{1/2}}{\frac{1}{2}} + c$$

$$\therefore \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$= \sqrt{x^2 + 2bx + c} + c$$

Ans.

**Q.7**  $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$  (Lhr. Board 2005, 2008)



**Solution:**

$$\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$$\begin{aligned}f(x) &= \tan x \\f'(x) &= \sec^2 x\end{aligned}$$

$$= \int (\tan x)^{-1/2} \cdot \sec^2 x dx$$

$$\therefore \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$= \frac{(\tan x)^{1/2}}{\frac{1}{2}} + c$$

$$= 2 \sqrt{\tan x} + c$$

Ans.

**Q.8 (a)** Show that  $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln(x + \sqrt{x^2 - a^2}) + c$

**(b)** Show that  $\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} x + \frac{x}{a} \sqrt{a^2 - x^2} + c$

**Solution:**

(a) Taking

$$\int \frac{dx}{\sqrt{(x^2 - a^2)}}$$

Put

$$x = a \sec\theta \Rightarrow \sec\theta = \frac{x}{a}$$

$$dx = a \sec\theta \tan\theta d\theta$$

$$= \int \frac{a \sec\theta \tan\theta d\theta}{\sqrt{a^2 \sec^2\theta - a^2}}$$

$$= \int \frac{a \sec\theta \tan\theta}{\sqrt{a^2(\sec^2\theta - 1)}} d\theta$$

$$= \int \frac{a \sec\theta \tan\theta}{a \sqrt{\tan^2\theta}} d\theta$$

$$= \int \frac{a \sec\theta \tan\theta}{\tan\theta} d\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln(\sec\theta + \tan\theta) + c_1$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right) + c_1$$

$$= \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c_1$$

$$= \ln(x + \sqrt{x^2 - a^2}) - \ln(a) + c_1$$

$$= \ln(x + \sqrt{x^2 - a^2}) + c \text{ where } c = -\ln a + c_1$$

Hence proved.

(b) Taking

$$\int \sqrt{a^2 - x^2} dx$$

$$\text{Put } x = a \sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$dx = a \cos\theta d\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$$

$$= \int \sqrt{a^2 - a^2 \sin^2\theta} \cdot a \cos\theta d\theta$$

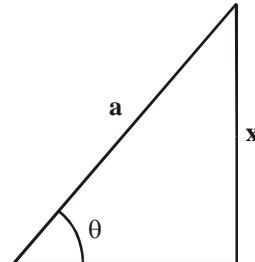
$$= \int \sqrt{a^2(1 - \sin^2\theta)} \cdot a \cos\theta d\theta$$



$$\begin{aligned}
 x^2 &= a^2 + P^2 \\
 P^2 &= x^2 - a^2 \\
 P &= \sqrt{x^2 - a^2} \quad \left| \begin{array}{l} \tan\theta = \frac{P}{a} \\ = \frac{\sqrt{x^2 - a^2}}{a} \end{array} \right.
 \end{aligned}$$

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$$\begin{aligned}
 &= \int a \sqrt{\cos^2 \theta} \cdot a \cos \theta d\theta \\
 &= a^2 \int \cos \theta \cdot \cos d\theta \\
 &= a^2 \int \cos \theta d\theta \\
 &= a^2 \int \frac{1 + \cos 2\theta}{2} d\theta \quad \left( \because \cos 2\theta = 2\cos^2 \theta - 1 \right. \\
 &= \frac{a^2}{2} \int (1 + \cos 2\theta) d\theta \quad \left. \begin{array}{l} 2\cos^2 \theta = 1 + \cos 2\theta \\ \cos^2 \theta = \frac{1+\cos 2\theta}{2} \end{array} \right) \\
 &= \frac{a^2}{2} \int d\theta + \frac{a^2}{2} \int \cos 2\theta d\theta \\
 &= \frac{a^2}{2} \theta + \frac{a^2}{2} \cdot \frac{\sin 2\theta}{2} + C \\
 &= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{a^2}{4} \cdot 2 \sin \theta \cos \theta + C \\
 &= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{a^2}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + C \\
 &= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C
 \end{aligned}$$



$a^2 = B^2 + x^2$ $B^2 = a^2 - x^2$ $B = \sqrt{a^2 - x^2}$	$\cos \theta = \frac{B}{a}$ $= \frac{\sqrt{a^2 - x^2}}{a}$ $\sin \theta = \frac{x}{a}$
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Hence proved

Evaluate the following integrals

Q.9  $\int \frac{dx}{(1+x^2)^{3/2}}$

**Solution:**

$$\int \frac{dx}{(1+x^2)^{3/2}}$$

Put  $x = \tan \theta$

$$dx = \sec^2 \theta d\theta$$

$$= \int \frac{\sec^2 \theta d\theta}{(1 + \tan^2 \theta)^{3/2}} = \int \frac{\sec^2 \theta}{(\sec^2 \theta)^{3/2}} d\theta$$

$$= \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta$$

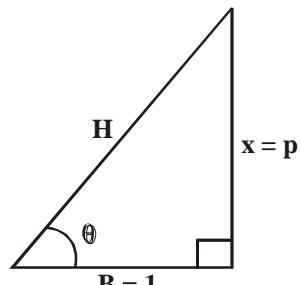
$$= \int \frac{d\theta}{\sec \theta} d\theta$$

$$= \int \cos \theta d\theta$$

$$= \sin \theta + c$$

$$\because \sin \theta = \frac{P}{H}$$

$$\sin \theta = \frac{x}{\sqrt{1+x^2}}$$



$$H^2 = 1 + x^2$$

$$H = \sqrt{1 + x^2}$$

$$Q.10 \quad \int \frac{1}{(1+x^2) \tan^{-1} x} dx$$

**Solution:**

$$\int \frac{1}{(1+x^2) \tan^{-1} x} dx$$

$$= \int \frac{1}{\tan^{-1} x} dx$$

$$= \ln(\tan^{-1} x) + c \quad \text{Ans.}$$

$$\because f(x) = \tan^{-1} x$$

$$\therefore f'(x) = \frac{1}{1+x^2}$$

$$Q.11 \quad \int \sqrt{\frac{1+x}{1-x}} dx$$

**Solution:**

$$\int \sqrt{\frac{1+x}{1-x}} dx$$

$$= \int \sqrt{\frac{1+x}{1-x} \times \frac{1+x}{1+x}} dx$$

$$= \int \sqrt{\frac{(1+x)^2}{1-x^2}} dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}} dx \quad \because \text{L.C.M Breaking}$$

$$\begin{aligned}
 &= \int \left( \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \right) dx \\
 &= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2} \int (1-x^2)^{-1/2} \cdot -2x dx \\
 &= \sin^{-1} x - \frac{1}{2} \frac{(1-x^2)^{1/2}}{\frac{1}{2}} + c \\
 &= \sin^{-1} x - \sqrt{1-x^2} + c \quad \text{Ans.}
 \end{aligned}$$

**Q.12**  $\int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta$

**Solution:**

$$\int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta$$

Put

$$\cos \theta = t$$

$$-\sin \theta d\theta = dt$$

$$d\theta = \frac{dt}{-\sin \theta}$$



$$\begin{aligned}
 &= \int \frac{\sin \theta}{1 + t^2} \times \frac{dt}{-\sin \theta} \\
 &= - \int \frac{dt}{1 + t^2} \\
 &= - \tan^{-1}(t) + c \\
 &= - \tan^{-1}(\cos \theta) + c \quad \text{Ans.} \quad \because t = \cos \theta
 \end{aligned}$$

**Q.13**  $\int \frac{ax}{\sqrt{a^2 - x^4}} dx$

**Solution:**

$$\int \frac{ax}{\sqrt{a^2 - x^4}} dx$$

$$= \int \frac{ax}{\sqrt{a^2 - (x^2)^2}} dx$$

Put

$$x^2 = t$$

$$2x dx = dt$$

$$dx = \frac{dt}{2x}$$

$$= \int \frac{ax}{\sqrt{a^2 - t^2}} \times \frac{dt}{2x}$$

$$= \frac{a}{2} \int \frac{dt}{\sqrt{a^2 - t^2}}$$

$$= \frac{a}{2} \sin^{-1} \left( \frac{t}{a} \right) + c$$

$$= \frac{a}{2} \sin^{-1} \left( \frac{x^2}{a} \right) + c \quad \text{Ans.} \quad \because t = x^2$$

$$\text{Q.14} \quad \int \frac{dx}{\sqrt{7 - 6x - x^2}}$$



**Solution:**

$$\int \frac{dx}{\sqrt{7 - 6x - x^2}}$$

$$= \int \frac{dx}{\sqrt{-(x^2 + 6x - 7)}}$$

$$= \int \frac{dx}{\sqrt{-(x^2 + 6x + 9 - 9 - 7)}}$$

$$= \int \frac{dx}{\sqrt{-(x+3)^2 - 16}}$$

$$\begin{aligned}
 &= \int \frac{dx}{\sqrt{16 - (x+3)^2}} \\
 &= \int \frac{dx}{\sqrt{(4)^2 - (x+3)^2}} \quad \because \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C \\
 &= \sin^{-1}\left(\frac{x+3}{4}\right) + C \quad \text{Ans.}
 \end{aligned}$$

**Q.15**  $\int \frac{\cos x}{\sin x \ln \sin x} dx$  (Guj. Board 2008)

**Solution:**

$$\int \frac{\cos x}{\sin x \ln \sin x} dx$$

Put

$$\begin{aligned}
 \ln \sin x &= t \\
 \frac{1}{\sin x} \cos x dx &= dt \\
 dx &= \frac{\sin x}{\cos x} dt \\
 &= \int \frac{\cos x}{\sin x t} \times \frac{\sin x}{\cos x} dt \\
 &= \int \frac{dt}{t} \\
 &= \ln t + c \\
 &= \ln (\ln \sin x) + c \quad \text{Ans.} \quad \because t = \ln \sin x
 \end{aligned}$$

**Q.16**  $\int \cos x \left( \frac{\ln \sin x}{\sin x} \right) dx$  (Lhr. Board 2005)

**Solution:**

$$\int \cos x \left( \frac{\ln \sin x}{\sin x} \right) dx$$

Put

$$\ln \sin x = t$$

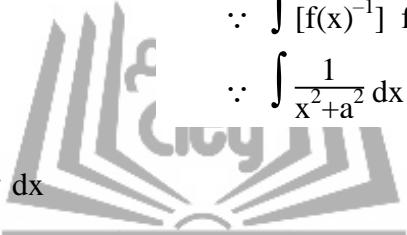
$$\begin{aligned}
 \frac{1}{\sin x} \cos x \cdot dx &= dt \\
 dx &= \frac{\sin x}{\cos x} dt \\
 &= \int \cos x \left( \frac{t}{\sin x} \right) \cdot \frac{\sin x}{\cos x} dt \\
 &= \int t dt \\
 &= \frac{t^2}{2} + c \\
 &= \frac{(\ln \sin x)^2}{2} + c \quad \text{Ans.} \quad \because t = \ln \sin x
 \end{aligned}$$

**Q.17**  $\int \frac{x dx}{4 + 2x + x^2}$

Formula used

**Solution:**

$$\begin{aligned}
 &\int \frac{x dx}{4 + 2x + x^2} \quad \because \int [f(x)^{-1}] f'(x) dx = \ln f(x) + c \\
 &\quad \because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + c \\
 &= \frac{1}{2} \int \frac{2x + 2 - 2}{4 + 2x + x^2} dx \\
 &= \frac{1}{2} \int \left( \frac{2x + 2}{4 + 2x + x^2} - \frac{2}{4 + 2x + x^2} \right) dx \\
 &= \frac{1}{2} \int \frac{2x + 2}{4 + 2x + x^2} dx - \frac{2}{2} \int \frac{dx}{4 + 2x + x^2} \\
 &= \frac{1}{2} \ln(x^2 + 2x + 4) - \int \frac{dx}{x^2 + 2x + 1 - 1 + 4} \\
 &= \frac{1}{2} \ln(x^2 + 2x + 4) - \int \frac{dx}{(x + 1)^2 + 3} \\
 &= \frac{1}{2} \ln(x^2 + 2x + 4) - \int \frac{dx}{(x + 1)^2 + (\sqrt{3})^2} \\
 &= \frac{1}{2} \ln(x^2 + 2x + 4) - \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x + 1}{\sqrt{3}} \right) + c \quad \text{Ans.}
 \end{aligned}$$



$$\text{Q.18} \quad \int \frac{x}{x^4 + 2x^2 + 5} dx$$

**Solution:**

$$\begin{aligned}
 & \int \frac{x}{x^4 + 2x^2 + 5} dx \\
 &= \int \frac{x}{(x^2)^2 + 2x^2 + 5} dx \\
 &\text{Put } x^2 = t \\
 &\quad 2x dx = dt \\
 &\quad dx = \frac{dt}{2x} \\
 &= \int \frac{x}{t^2 + 2t + 5} \times \frac{dt}{2x} \\
 &= \frac{1}{2} \int \frac{dt}{t^2 + 2t + 1 - 1 + 5} \\
 &= \frac{1}{2} \int \frac{dt}{(t+1)^2 + 4} \\
 &= \frac{1}{2} \int \frac{dt}{(t+1)^2 + (2)^2} \\
 &= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \left( \frac{t+1}{2} \right) + c \\
 &= \frac{1}{4} \tan^{-1} \left( \frac{x^2 + 1}{2} \right) + c \quad \text{Ans.}
 \end{aligned}$$

$$\because t = x^2$$

$$\text{Q.19} \quad \int \left[ \cos \left( \sqrt{x} - \frac{x}{2} \right) \right] \times \left( \frac{1}{\sqrt{x}} - 1 \right) dx$$

**Solution:**

$$\begin{aligned}
 & \int \left[ \cos \left( \sqrt{x} - \frac{x}{2} \right) \right] \times \left( \frac{1}{\sqrt{x}} - 1 \right) dx \\
 &= 2 \int \cos \left( \sqrt{x} - \frac{x}{2} \right) \times \frac{1}{2} \left( \frac{1}{\sqrt{x}} - 1 \right) dx
 \end{aligned}$$

$$\begin{aligned}
 \because f(x) &= \sqrt{x} - \frac{x}{2} \\
 f'(x) &= \frac{1}{2\sqrt{x}} - \frac{1}{2} \\
 f'(x) &= \frac{1}{2} \left( \frac{1}{\sqrt{x}} - 1 \right)
 \end{aligned}$$

$$= 2 \int \cos(\sqrt{x} - \frac{x}{2}) \times \left( \frac{1}{2\sqrt{x}} - \frac{1}{2} \right) dx$$

$$= 2 \sin(\sqrt{x} - \frac{x}{2}) + c \quad \text{Ans.}$$

**Q.20**  $\int \frac{x+2}{\sqrt{x+3}} dx$

**Solution:**

$$\int \frac{x+2}{\sqrt{x+3}} dx$$

Put

$$\sqrt{x+3} = t$$

$$x+3 = t^2 \Rightarrow x = t^2 - 3$$

$$dx = 2t dt$$

$$= \int \frac{t^2 - 3 + 2}{t} \times 2t dt$$

$$= 2 \int (t^2 - 1) dt$$

$$= 2 \int t^2 dt - 2 \int dt$$

$$= \frac{2t^3}{3} - 2t + c$$

$$= \frac{2}{3} (x+3)^{\frac{3}{2}} - 2\sqrt{x+3} + c \quad \text{Ans.} \quad \therefore t = \sqrt{x+3}$$

**Q.21**  $\int \frac{\sqrt{2}}{\sin x + \cos x} dx \quad (\text{Lhr. Board 2008})$

**Solution:**

$$\int \frac{\sqrt{2}}{\sin x + \cos x} dx$$

$$= \int \frac{1}{\frac{1}{\sqrt{2}} (\sin x + \cos x)} dx$$

$$\begin{aligned}
 &= \int \frac{dx}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} \\
 &= \int \frac{dx}{\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4}} \quad \boxed{\because \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta} \\
 &= \int \frac{dx}{\cos(x - \frac{\pi}{4})} \\
 &= \int \sec\left(x - \frac{\pi}{4}\right) dx \\
 &= \ln \left| \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right| + c \quad \text{Ans.}
 \end{aligned}$$

**Q.22**  $\int \frac{dx}{\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x}$



**Solution:**

$$\begin{aligned}
 &\int \frac{dx}{\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x} \\
 &= \int \frac{dx}{\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3}} \quad \boxed{\because \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta} \\
 &= \int \frac{dx}{\sin\left(x + \frac{\pi}{3}\right)} \\
 &= \int \csc\left(x + \frac{\pi}{3}\right) dx \\
 &= \ln \left| \csc\left(x + \frac{\pi}{3}\right) - \cot\left(x + \frac{\pi}{3}\right) \right| + c \quad \text{Ans.}
 \end{aligned}$$