

EXERCISE 3.4

Q.2 Evaluate the following integrals by parts add a word representing all the functions are defined.

(i) $\int x \sin x \, dx$

(ii) $\int \ln x \, dx$

(iii) $\int x \ln x \, dx$

(iv) $\int x^2 \ln x \, dx$

(v) $\int x^3 \ln x \, dx$

(vi) $\int x^4 \ln x \, dx$

(vii) $\int \tan^{-1} x \, dx$

(viii) $\int x^2 \sin x \, dx$

(ix) $\int x^2 \tan^{-1} x \, dx$

(x) $\int x \tan^{-1} x \, dx$

(xi) $\int x^3 \tan^{-1} x \, dx$

(xii) $\int x^3 \cos x \, dx$

(xiii) $\int \sin^{-1} x \, dx$

(xiv) $\int x \sin^{-1} x \, dx$

(xv) $\int e^x \sin x \cos x \, dx$

(xvi) $\int x \sin x \cos x \, dx$

(xvii) $\int x \cos^2 x \, dx$

(xviii) $\int x \sin^2 x \, dx$

(xix) $\int (\ln x)^2 \, dx$

(xx) $\int \ln(\tan x) \sec^2 x \, dx$

(xxi) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$

Solution:

(i) $\int x \sin x \, dx$

$$= x \int \sin x \, dx - \int \left[\int \sin x \, dx \cdot \frac{d}{dx}(x) \right] dx$$

 $\because x \rightarrow 1^{\text{st}}$ function
 $\sin x \rightarrow 2^{\text{nd}}$ function

$$= -x \cos x - \int -\cos x \cdot 1 \, dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + c$$

Ans.

(ii) $\int \ln x \, dx$ (Guj. Board 2006) (Lhr. Board 2006)

$$= \int \ln x \cdot 1 \, dx$$

 $\because \ln x \rightarrow 1^{\text{st}}$ function
 $1 \rightarrow 2^{\text{nd}}$ function

$$\begin{aligned}
 &= \ln x \int dx - \int \left[\int dx \cdot \frac{d}{dx} (\ln x) \right] dx \\
 &= \ln x \cdot x - \int x \cdot \frac{1}{x} dx \\
 &= x \ln x - \int dx \\
 &= x \ln x - x + c \quad \text{Ans.}
 \end{aligned}$$

(iii) $\int x \ln x dx$ (Lhr. Board 2006, 2007, 2008)

$$\begin{aligned}
 &= \ln x \int x dx - \int \left[\int x dx \cdot \frac{d}{dx} (\ln x) \right] dx \\
 &= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\
 &= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x dx \\
 &= \frac{x^2 \ln x}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + c \\
 &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c \\
 &= \frac{1}{2} x^2 (\ln x - \frac{1}{2}) + c \quad \text{Ans.}
 \end{aligned}$$

(iv) $\int x^2 \ln x dx$

$$\begin{aligned}
 &= \ln x \int x^2 dx - \int \left[\int x^2 dx \cdot \frac{d}{dx} (\ln x) \right] dx \\
 &= \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx \\
 &= \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^2 dx \\
 &= \frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + c \\
 &= \frac{x^3}{3} (\ln x - \frac{1}{3}) + c \quad \text{Ans.}
 \end{aligned}$$

(v) $\int x^3 \ln x dx$ (Lhr. Board 2007)

$\because \ln x \rightarrow 1^{\text{st}}$ function
 $x^3 \rightarrow 2^{\text{nd}}$ function

$$\begin{aligned}
 &= \ln x \int x^3 dx - \int \left[\int x^3 dx \cdot \frac{d}{dx} (\ln x) \right] dx \\
 &= \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx \\
 &= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 dx \\
 &= \frac{x^4 \ln x}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + c \\
 &= \frac{x^4}{4} \left(\ln x - \frac{1}{4} \right) + c \quad \text{Ans.}
 \end{aligned}$$

(vi) $\int x^4 \ln x dx$

$$\begin{aligned}
 &= \ln x \int x^4 dx - \int \left[\int x^4 dx \cdot \frac{d}{dx} (\ln x) \right] dx \quad \boxed{\begin{array}{lcl} \because \ln x & \rightarrow & 1^{\text{st}} \text{ function} \\ x^4 & \rightarrow & 2^{\text{nd}} \text{ function} \end{array}} \\
 &= \ln x \cdot \frac{x^5}{5} - \int \frac{x^5}{5} \cdot \frac{1}{x} dx \\
 &= \frac{x^5 \ln x}{5} - \frac{1}{5} \int x^4 dx \\
 &= \frac{x^5 \ln x}{5} - \frac{1}{5} \cdot \frac{x^5}{5} + c \\
 &= \frac{x^5}{5} \left(\ln x - \frac{1}{5} \right) + c \quad \text{Ans.}
 \end{aligned}$$

(vii) $\int \tan^{-1} x dx$ (Lhr. Board 2011) (Guj. Board 2008)

$$\begin{aligned}
 &= \int \tan^{-1} x \cdot 1 dx \quad \boxed{\begin{array}{lcl} \because \tan^{-1} x & \rightarrow & 1^{\text{st}} \text{ function} \\ x & \rightarrow & 2^{\text{nd}} \text{ function} \end{array}} \\
 &= \tan^{-1} x \int dx - \int \left[\int dx \cdot \frac{d}{dx} (\tan^{-1} x) \right] dx \\
 &= \tan^{-1} x \cdot x - \int x \cdot \frac{1}{1+x^2} dx \quad \boxed{\begin{array}{l} \because f(x) = 1+x^2 \\ \text{derivative} = 2x \end{array}} \\
 &= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx
 \end{aligned}$$

$$= \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c \quad \text{Ans.}$$

(viii) $\int x^2 \sin x \, dx$

$$\begin{aligned}
 &= x^2 \int \sin x \, dx - \int \left[\sin x \, dx \cdot \frac{d}{dx}(x^2) \right] \, dx \\
 &= x^2 (-\cos x) - \int -\cos x \cdot 2x \, dx \\
 &= -x^2 \cos x + 2 \int x \cos x \, dx \\
 &= -x^2 \cos x + 2 \left[x \sin x \, dx - \int \left(\int \sin x \, dx \frac{d}{dx}(x) \right) \, dx \right] \\
 &= -x^2 \cos x + 2 \left[x \sin x - \int \sin x \, dx \right] \\
 &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c \\
 &= -x^2 \cos x + 2x \sin x + 2\cos x + c \quad \text{Ans.}
 \end{aligned}$$

(ix) $\int x^2 \tan^{-1} x \, dx$ (Lhr. Board 2006, 2008)

$$\begin{aligned}
 &= \tan^{-1} x \int x^2 \, dx - \int \left[x^2 \, dx \cdot \frac{d}{dx}(\tan^{-1} x) \right] \, dx \\
 &= \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} \, dx \\
 &= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx \quad \because 1+x^2 \sqrt{\frac{x}{x^3}} \\
 &= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) \, dx \quad \underline{\underline{\frac{x^3}{1+x^2}}} \\
 &= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int x \, dx + \frac{1}{3} \int \frac{2x}{1+x^2} \, dx \\
 &\quad \because f(x) = 1+x^2 \\
 &\quad f'(x) = 2x
 \end{aligned}$$

$$\begin{aligned} &= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \frac{x^2}{2} + \frac{1}{6} \ln(1 + x^2) + c \\ &= \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{1}{6} \ln(1 + x^2) + c \quad \text{Ans.} \end{aligned}$$



$$(x) \int x \tan^{-1} x \, dx$$

$\because \tan^{-1} x \rightarrow 1^{\text{st}}$ function
 v v 2nd function

$$\begin{aligned}
 &= \tan^{-1} x \int x \, dx - \int \left[\int x \, dx \cdot \frac{d}{dx} (\tan^{-1} x) \right] dx \\
 &= \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx \\
 &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\
 &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\
 &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2} \\
 &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c \\
 &= \left(\frac{1}{2} \tan^{-1} x \right) (x^2 + 1) - \frac{1}{2} x + c \quad \text{Ans.}
 \end{aligned}$$

$$(xi) \int x^3 \tan^{-1} x \, dx \quad (\text{Guj. Board 2007})$$

$\because \tan^{-1} x \rightarrow 1^{\text{st}}$ function
 ..³ v v 2nd function

$$\begin{aligned}
 &= \tan^{-1} x \int x^3 \, dx - \int \left[x^3 \, dx \cdot \frac{d}{dx} (\tan^{-1} x) \right] dx \\
 &= \tan^{-1} x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} dx \\
 &= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \int \frac{x^4}{1+x^2} dx \\
 &= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx \\
 &= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \int x^2 \, dx + \frac{1}{4} \int dx - \frac{1}{4} \int \frac{dx}{1+x^2} \\
 &= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \cdot \frac{x^3}{3} + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c \\
 &= \frac{1}{4} \left[(x^4 - 1) \tan^{-1} x - \frac{x^3}{3} + x \right] + c \quad \text{Ans.}
 \end{aligned}$$

(xii) $\int x^3 \cos x \, dx$

$\because x^3 \rightarrow 1^{\text{st}}$ function
 $\cos x \rightarrow 2^{\text{nd}}$ function

$$\begin{aligned}
 &= x^3 \int \cos x \, dx - \int [\cos x \, dx \cdot \frac{d}{dx}(x^3)] \, dx \\
 &= x^3 \sin x - \int \sin x \cdot 3x^2 \, dx \\
 &= x^3 \sin x - 3 \int x^2 \sin x \, dx \\
 &= x^3 \sin x - 3 \left[x^2 \int \sin x \, dx - \int \left(\sin x \, dx \cdot \frac{d}{dx}(x^2) \right) \, dx \right] \\
 &= x^3 \sin x - 3x^2 (-\cos x) + 3 \int -\cos x \cdot 2x \, dx \\
 &= x^3 \sin x + 3x^2 \cos x - 6 \int x \cos x \, dx \\
 &= x^3 \sin x + 3x^2 \cos x - 6 \left[x \int \cos x \, dx - \int (\int \cos x \, dx \cdot \frac{d}{dx}(x)) \, dx \right] \\
 &= x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6 \int \sin x \, dx \\
 &= x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6(-\cos x) + c \\
 &= (x^3 - 6x) \sin x + (3x^2 - 6) \cos x + c \quad \text{Ans.}
 \end{aligned}$$

(xiii) $\int \sin^{-1} x \, dx$

$\because \sin^{-1} x \rightarrow 1^{\text{st}}$ function
 $1 \rightarrow 2^{\text{nd}}$ function

$$\begin{aligned}
 &= \int \sin^{-1} x \cdot 1 \, dx \\
 &= \sin^{-1} x \int dx - \int \left[\int dx \cdot \frac{d}{dx}(\sin^{-1} x) \right] \, dx \\
 &= \sin^{-1} x \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx \\
 &= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-1/2} \cdot -2x \, dx \\
 &= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} + c \\
 &= x \sin^{-1} x + \sqrt{1-x^2} + c \quad \text{Ans.}
 \end{aligned}$$

$$(xiv) \int x \sin^{-1} x dx$$

$$\begin{aligned}
 &= \sin^{-1} x \int x dx - \int \left[[x dx] \cdot \frac{d}{dx} (\sin^{-1} x) \right] dx \\
 &= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx \\
 &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx \\
 &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}} \\
 &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \sin^{-1} x \\
 &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} I - \frac{1}{2} \sin^{-1} x
 \end{aligned}$$

∵ $\sin^{-1} x$ → 1st function

 1 2nd function

(1)

Taking

$$I = \int \sqrt{1-x^2} dx$$

Put

$$x = \sin \theta \Rightarrow \theta = \sin^{-1} x$$

$$dx = \cos \theta d\theta$$

$$I = \int \sqrt{1-\sin^2 \theta} \cdot \cos \theta d\theta$$

$$I = \int \sqrt{\cos^2 \theta} \cdot \cos \theta d\theta$$

$$I = \int \cos \theta \cdot \cos \theta d\theta$$

$$I = \int \cos^2 \theta d\theta$$

$$I = \int \frac{1+\cos 2\theta}{2} d\theta$$

$\left(\because \cos^2 \theta = \frac{1+\cos 2\theta}{2}, \quad 2\cos^2 \theta = 1 + \cos 2\theta, \quad \cos^2 \theta = \frac{1+\cos 2\theta}{2} \right)$

$$I = \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

$$I = \frac{1}{2} \int d\theta + \frac{1}{2.2} \int \cos 2\theta \cdot 2d\theta$$

$$I = \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{4} \cdot 2 \sin \theta \cos \theta + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{\cos^2 \theta} + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1 - \sin^2 \theta} + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1 - x^2} + c$$

\therefore From equation (1)

$$\begin{aligned} &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left[\frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1 - x^2} \right] - \frac{1}{2} \sin^{-1} x + c \\ &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4} \sin^{-1} x + \frac{x \sqrt{1 - x^2}}{4} - \frac{1}{2} \sin^{-1} x + c \\ &= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4} \sin^{-1} x + \frac{x \sqrt{1 - x^2}}{4} + c \quad \text{Ans.} \end{aligned}$$

$$(xv) \int e^x \sin x \cos x \, dx$$

$$= \frac{1}{2} \int e^x 2 \sin x \cos x \, dx$$

$$= \frac{1}{2} \int e^x \sin 2x \, dx$$

$$= \frac{1}{2} I \quad \text{_____} \quad (1)$$

Where

$$I = e^x \sin 2x \, dx$$

$\because e^x \rightarrow 1^{\text{st}}$ function
 $\sin 2x \rightarrow 2^{\text{nd}}$ function

$$= e^x \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot e^x \, dx$$

$$= -\frac{e^x \cos 2x}{2} + \frac{1}{2} \int e^x \cos 2x \, dx$$

$$= -\frac{e^x \cos 2x}{2} + \frac{1}{2} \left[e^x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot e^x dx \right]$$

$$I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} - \frac{1}{4} \int e^x \sin 2x dx$$

$$I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} - \frac{1}{4} I + c$$

$$I + \frac{I}{4} = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} + c$$

$$\frac{5I}{4} = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} + c$$

$$I = -\frac{4e^x \cos 2x}{10} + \frac{4e^x \sin 2x}{20} + c$$

$$I = -\frac{2e^x \cos 2x}{5} + \frac{e^x \sin 2x}{5} + c$$

$\because e^x \rightarrow 1^{\text{st}}$ function
 $\cos 2x \rightarrow 2^{\text{nd}}$ function

From equation (1)

$$\begin{aligned} &= \frac{1}{2} \left(-\frac{2e^x \cos 2x}{5} + \frac{e^x \sin 2x}{5} \right) + c \\ &= \frac{1}{5} e^x \left(-2\cos 2x + \frac{\sin 2x}{2} \right) + c \\ &= \frac{1}{5} e^x \left[-(1 - 2\sin^2 x) + \frac{\sin 2x}{2} \right] + c \\ &= \frac{1}{5} e^x (2\sin^2 x - 1 + \frac{1}{2} \sin 2x) + c \quad \text{Ans.} \end{aligned}$$

(xvi) $\int x \sin x \cos x dx$

$$= \frac{1}{2} \int x \cdot 2 \sin x \cos x dx$$

$$= \frac{1}{2} \int x \sin 2x dx$$

$$= \frac{1}{2} [x \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot 1 dx]$$

$\because x \rightarrow 1^{\text{st}}$ function
 $\sin 2x \rightarrow 2^{\text{nd}}$ function

$$= \frac{-x \cos 2x}{4} + \frac{1}{4} \int \cos 2x dx$$

$$= \frac{-x \cos 2x}{4} + \frac{1}{4} \cdot \frac{\sin 2x}{2} + c$$

$$= \frac{1}{4} (-x \cos 2x + \frac{2 \sin x \cos x}{2}) + c$$

$$= \frac{1}{4} (\sin x \cos x - x \cos 2x) + c \quad \text{Ans.}$$

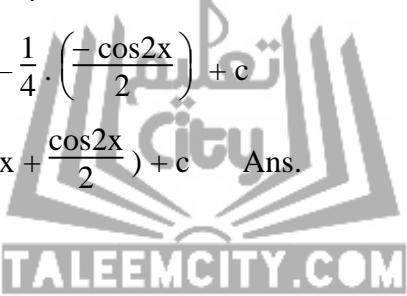
(xvii) $\int x \cos^2 x \, dx$

$$\begin{aligned}
 &= \int x \left(\frac{1 + \cos 2x}{2} \right) \, dx \\
 &= \frac{1}{2} \int (x + x \cos 2x) \, dx \\
 &= \frac{1}{2} \int x \, dx + \frac{1}{2} \int x \cos 2x \, dx \\
 &= \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot 1 \, dx \right] \\
 &= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \int \sin 2x \, dx \\
 &= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \cdot \left(\frac{-\cos 2x}{2} \right) + c \\
 &= \frac{1}{4} \left(x^2 + x \sin 2x + \frac{\cos 2x}{2} \right) + c \quad \text{Ans.}
 \end{aligned}$$

(xviii) $\int x \sin^2 x \, dx$

$$\begin{aligned}
 &= \int x \left(\frac{1 - \cos 2x}{2} \right) \, dx \quad \because \cos 2x = 1 - 2\sin^2 x \\
 &= \frac{1}{2} \int (x - x \cos 2x) \, dx \quad 2\sin^2 x = 1 - \cos 2x \\
 &= \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx \quad \sin^2 x = \frac{1 - \cos 2x}{2} \\
 &= \frac{1}{2} \cdot \frac{x^2}{2} - \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot 1 \, dx \right] \\
 &= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4} \int \sin 2x \, dx \\
 &= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4} \left[\frac{-\cos 2x}{2} \right] + c
 \end{aligned}$$

$\because x \rightarrow 1^{\text{st}}$ function
 $\cos 2x \rightarrow 2^{\text{nd}}$ function



$$\begin{aligned}
 &= \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + c \\
 &= \frac{1}{4} (x^2 - x \sin 2x - \frac{1}{2} \cos 2x) + c \quad \text{Ans.}
 \end{aligned}$$

(xix) $\int (\ln x)^2 dx$

$$= \int (\ln x)^2 \cdot 1 dx$$

 $\because (\ln x)^2 \rightarrow 1^{\text{st}}$ function
 $1 \rightarrow 2^{\text{nd}}$ function

$$\begin{aligned}
 &= (\ln x)^2 \int dx - \int \left[\int dx \cdot \frac{d}{dx} (\ln x)^2 \right] dx \\
 &= (\ln x)^2 \cdot x - \int x \cdot 2 (\ln x) \cdot \frac{1}{x} dx \\
 &= x (\ln x)^2 - 2 \int \ln x \cdot 1 dx \\
 &= x (\ln x)^2 - 2 \left[\ln x dx - \int \left(\int dx \cdot \frac{d}{dx} (\ln x) \right) dx \right] \\
 &= x (\ln x)^2 - 2 \ln x \cdot x + 2 \int x \cdot \frac{1}{x} dx \\
 &= x (\ln x)^2 - 2x \ln x + 2 \int dx \\
 &= x (\ln x)^2 - 2x \ln x + 2x + c \\
 &= x \ln x (\ln x - 2) + 2x + c \quad \text{Ans.}
 \end{aligned}$$

(xx) $\int \ln(\tan x) \sec^2 x dx$

$$\begin{aligned}
 &= \ln(\tan x) \int \sec^2 x dx - \int \left[\int \sec^2 x dx \cdot \frac{d}{dx} \ln(\tan x) \right] dx \\
 &= \ln(\tan x) \cdot \tan x - \int \tan x \cdot \frac{1}{\tan x} \cdot \sec^2 x dx \\
 &= \tan x \ln(\tan x) - \int \sec^2 x dx \\
 &= \tan x \ln(\tan x) - \tan x + c \quad \text{Ans.}
 \end{aligned}$$

Alternate Method**Solution:**

$$\begin{aligned}
 t &= \tan x \\
 dt &= \sec^2 x dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int \ln t \cdot 1 dt \\
 &= \ln t \cdot t - \int t \cdot \frac{1}{t} dt \\
 &= t \cdot \ln t - \int dt \\
 &= t \cdot \ln t - t + c \\
 &= \tan x \ln(\tan x) - \tan x + c
 \end{aligned}$$

(xxi) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Put

$$\begin{aligned}
 \sin^{-1} x &= t & \Rightarrow x &= \sin t \\
 \frac{1}{\sqrt{1-x^2}} dx &= dt \\
 dx &= \sqrt{1-x^2} dt \\
 &= \frac{\sin t \times t}{\sqrt{1-x^2}} \times \sqrt{1-x^2} dt \\
 &= t \sin t dt \\
 &= t \int \sin t dt - \int \left[\int \sin t dt \cdot \frac{d}{dt}(t) \right] dt \\
 &= t \cdot -\cos t - \int -\cos t \cdot 1 dt \\
 &= -t \cos t + \sin t + c \\
 &= -\sin^{-1} x \sqrt{\cos^2 t} + x + c \\
 &= x - \sin^{-1} x \sqrt{1 - \sin^2 t} + c \\
 &= x - \sqrt{1-x^2} \sin^{-1} x + c \quad \text{Ans.}
 \end{aligned}$$

$\because t \rightarrow 1^{\text{st}}$ function
 $\sin t \rightarrow 2^{\text{nd}}$ function

$\because t = \sin^{-1} x$

Q.2 Evaluate the following integral.

- | | | |
|-----------------------------------|--------------------------|--|
| (i) $\int \tan^4 x dx$ (L.B 2009) | (ii) $\int \sec^4 x dx$ | (iii) $\int e^x \sin 2x \cdot \cos x dx$ |
| (iv) $\int \tan^3 x \sec x dx$ | (v) $\int x^3 e^{5x} dx$ | (vi) $\int e^{-x} \sin 2x dx$ |

$$(vii) \int e^{2x} \cos 3x dx \quad (viii) \int \cosec^3 x dx$$

Solution:

$$(i) \int \tan^4 x dx$$

$$\begin{aligned} &= \int \tan^2 x \cdot \tan^2 x dx \\ &= \int \tan^2 x (\sec^2 x - 1) dx \\ &= \int (\tan^2 x \sec^2 x - \tan^2 x) dx \\ &= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1) dx \\ &= \frac{\tan^3 x}{3} - \int \sec^2 x dx + \int dx \\ &= \frac{\tan^3 x}{3} - \tan x + x + c \quad \text{Ans.} \end{aligned}$$

$$(ii) \int \sec^4 x dx$$

$$\begin{aligned} &= \int \sec^2 x \cdot \sec^2 x dx \\ &= \int \sec^2 x (1 + \tan^2 x) dx \\ &= \int (\sec^2 x + \sec x^2 \tan^2 x) dx \\ &= \int \sec^2 x dx + \int \tan^2 x \sec^2 x dx \\ &= \tan x + \frac{\tan^3 x}{3} + c \quad \text{Ans.} \end{aligned}$$

$$(iii) \int e^x \sin 2x \cos x dx$$

$$\begin{aligned} &= \frac{1}{2} \int e^x 2 \sin 2x \cos x dx \quad \boxed{\because 2 \sin \alpha \cos \beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)} \\ &= \frac{1}{2} \int e^x [\sin(2x+x) + \sin(2x-x)] dx \\ &= \frac{1}{2} \int (e^x \sin 3x + e^x \sin x) dx \\ &= \frac{1}{2} \int e^x \sin 3x dx + \frac{1}{2} \int e^x \sin x dx \\ &= \frac{1}{2} I_1 + \frac{1}{2} I_2 \quad \text{———} \quad (2) \end{aligned}$$



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$$I_1 = \int e^x \sin 3x dx$$

$$I_1 = e^x \int \sin 3x dx - \int \left[\int \sin 3x dx \cdot \frac{d}{dx} (e^x) \right] dx$$

$$I_1 = e^x \left(\frac{-\cos 3x}{3} \right) - \int \frac{-\cos 3x}{3} \cdot e^x dx$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{1}{3} \int e^x \cos 3x dx$$

$$I_1 = \frac{e^x \cos 3x}{3} + \frac{1}{3} \left[e^x \int \cos 3x dx - \int \left(\int \cos 3x dx \cdot \frac{d}{dx} (e^x) dx \right) \right]$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{1}{3} e^x \cdot \frac{\sin 3x}{3} - \frac{1}{3} \int \frac{\sin 3x}{3} \cdot e^x dx$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} - \frac{1}{9} \int e^x \sin 3x dx$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} - \frac{1}{9} I_1 + c_1$$

$$I_1 + \frac{1}{9} I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} + c_1$$

$$\frac{10}{9} I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} + c_1$$

$$I_1 = -\frac{9e^x \cos 3x}{30} + \frac{9e^x \sin 3x}{90} + c_1$$

$$I_1 = -\frac{3e^x \cos 3x}{10} + \frac{-e^x \sin 3x}{10} + c_1$$

$$I_2 = \int e^x \sin x dx$$

$$I_2 = e^x \int \sin x dx - \int \left[\int \sin x dx \cdot \frac{d}{dx} (e^x) \right] dx$$

$$I_2 = e^x (-\cos x) - \int -\cos x \cdot e^x dx$$

$$I_2 = -e^x \cos x + \int e^x \cos x dx$$

$$I_2 = -e^x \cos x + e^x \int \cos x dx - \int \left[\int \cos x dx \cdot \frac{d}{dx} (e^x) \right] dx$$

$$I_2 = -e^x \cos x + e^x \sin x - \int \sin x e^x dx$$

$$I_2 = -e^x \cos x + e^x \sin x - I_2 + c_2$$

$$2I_2 = -e^x \cos x + e^x \sin x + c_2$$

$$I_2 = \frac{-e^x \cos x}{2} + \frac{e^x \sin x}{2} + c_2$$

∴ From eq. (1)

$$= \frac{1}{2} \left[-\frac{3e^x \cos 3x}{10} + \frac{e^x \sin 3x}{10} + c_1 \right] + \frac{1}{2} \left[\frac{-e^x \cos x}{2} + \frac{e^x \sin x}{2} + c_2 \right]$$

$$= \frac{-3e^x \cos 3x}{20} + \frac{e^x \sin 3x}{20} + c_1 + \frac{-e^x \cos x}{4} + \frac{e^x \sin x}{4} + c_2$$

$$= \frac{1}{4} e^x \left[\frac{1}{5} \sin 3x - \frac{3}{5} \cos 3x + \sin x - \cos x \right] + c \quad \text{where } c = c_1 + c_2$$

(iv) $\int \tan^3 x \sec x dx$

$$= \int \tan^2 x \cdot \sec x \tan x dx$$

$$= \int (\sec^2 x - 1) \sec x \tan x dx$$

$$= \int \sec^2 x \cdot \sec x \tan x dx - \int \sec x \tan x dx$$

$$= \frac{\sec^3 x}{3} - \sec x + c \quad \text{Ans.}$$

(v) $\int x^3 e^{5x} dx$

$$= x^3 \int e^{5x} dx - \int \left[\int e^{5x} dx \cdot \frac{d}{dx}(x^3) \right] dx$$

$$= x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 3x^2 dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} \int x^2 e^{5x} dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} \left[x^2 \int e^{5x} dx - \int \left(\int e^{5x} dx \cdot \frac{d}{dx}(x^2) \right) dx \right]$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} x^2 \cdot \frac{e^{5x}}{5} + \frac{3}{5} \int \frac{e^{5x}}{5} \cdot 2x dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25} \int x e^{5x} dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25} \left[x \int e^{5x} dx - \int \left(\int e^{5x} dx \cdot \frac{d}{dx}(x) \right) dx \right]$$

$$\begin{aligned}
 &= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25} x \cdot \frac{e^{5x}}{5} - \frac{6}{25} \int \frac{e^{5x}}{5} \cdot 1 dx \\
 &= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125} \int e^{5x} dx \\
 &= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125} \cdot \frac{e^{5x}}{5} + c \\
 &= \frac{1}{5} e^{5x} (x^3 - \frac{3}{5} x^2 + \frac{6x}{25} - \frac{6}{125}) + c \quad \text{Ans.}
 \end{aligned}$$

(vi) $\int e^{-x} \sin 2x \, dx$

Let

$$\begin{aligned}
 I &= \int e^{-x} \sin 2x \, dx \\
 I &= e^{-x} \int \sin 2x \, dx - \int \left[\int \sin 2x \, dx \cdot \frac{d}{dx}(e^{-x}) \right] dx \\
 I &= e^{-x} \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot -e^{-x} dx \\
 I &= -\frac{e^{-x} \cos 2x}{2} - \frac{1}{2} \int e^{-x} \cos 2x \, dx \\
 I &= -\frac{e^{-x} \cos 2x}{2} - \frac{1}{2} \left[e^{-x} \int \cos 2x \, dx - \int \left(\int \cos 2x \, dx \cdot \frac{d}{dx}(e^{-x}) \right) dx \right] \\
 I &= -\frac{e^{-x} \cos 2x}{2} - \frac{1}{2} e^{-x} \frac{\sin 2x}{2} + \frac{1}{2} \int \frac{\sin 2x}{2} (-e^{-x}) dx \\
 I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} - \frac{1}{4} \int e^{-x} \sin 2x \, dx \\
 I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} - \frac{1}{4} I + c \\
 I + \frac{1}{4} I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} + c \\
 \frac{5}{4} I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} + c \\
 I &= \frac{-4e^{-x} \cos 2x}{10} - \frac{4e^{-x} \sin 2x}{20} + c \\
 I &= -\frac{2e^{-x} \cos 2x}{5} - \frac{e^{-x} \sin 2x}{5} + c \\
 I &= -\frac{2}{5} e^{-x} (\cos 2x + \frac{\sin 2x}{2}) + c \quad \text{Ans.}
 \end{aligned}$$

$$(vii) \int e^{2x} \cos 3x dx$$

Let

$$I = \int e^{2x} \cos 3x dx$$

$$I = e^{2x} \int \cos 3x dx - \int \left[\int \cos 3x dx \cdot \frac{d}{dx}(e^{2x}) \right] dx$$

$$I = e^{2x} \cdot \frac{\sin 3x}{3} - \int \frac{\sin 3x}{3} \cdot e^{2x} \cdot 2dx$$

$$I = \frac{e^{2x} \sin 3x}{3} - \frac{2}{3} \int e^{2x} \sin 3x dx$$

$$I = \frac{e^{2x} \sin 3x}{3} - \frac{2}{3} \left[e^{2x} \int \sin 3x dx - \int \left(\int \sin 3x dx \cdot \frac{d}{dx}(e^{2x}) \right) dx \right]$$

$$I = \frac{e^{2x} \sin 3x}{3} - \frac{2}{3} e^{2x} \left(\frac{-\cos 3x}{3} \right) + \frac{2}{3} \int \frac{-\cos 3x}{3} \cdot 2e^{2x} dx$$

$$I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} - \frac{4}{9} \int e^{2x} \cos 3x dx$$

$$I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} - \frac{4}{9} I + c$$

$$I + \frac{4}{9} I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} + c$$

$$\frac{13}{9} I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} + c$$

$$I = \frac{9e^{2x} \sin 3x}{3 \times 13} + \frac{18e^{2x} \cos 3x}{13 \times 9} + c$$

$$I = \frac{3e^{2x} \sin 3x}{13} + \frac{2e^{2x} \cos 3x}{13} + c$$

$$I = \frac{3}{13} e^{2x} (\sin 3x + \frac{2}{3} \cos 3x) + c \quad \text{Ans.}$$

$$(viii) \int \operatorname{cosec}^3 x dx$$

Let

$$I = \int \operatorname{cosec}^3 x dx$$

$$I = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec} x dx$$

$$\begin{aligned}
 I &= \operatorname{cosec} x \int \operatorname{cosec}^2 x dx - \int [\int \operatorname{cosec}^2 x dx \cdot \frac{d}{dx} (\operatorname{cosec} x)] dx \\
 I &= \operatorname{cosec} x \cdot -\operatorname{cot} x - \int -\operatorname{cot} x \cdot -\operatorname{cosec} x \operatorname{cot} x dx \\
 I &= -\operatorname{cosec} \operatorname{cot} x - \int \operatorname{cosec} x \operatorname{cot}^2 x dx \\
 I &= -\operatorname{cosec} x \operatorname{cot} x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) dx \\
 I &= -\operatorname{cosec} x \operatorname{cot} x - \int \operatorname{cosec}^3 x dx + \int \operatorname{cosec} x dx \\
 I &= -\operatorname{cosec} x \operatorname{cot} x - I + \ln |\operatorname{cosec} x - \operatorname{cot} x| + c \\
 2I &= -\operatorname{cosec} x \operatorname{cot} x + \ln |\operatorname{cosec} x - \operatorname{cot} x| + c \\
 I &= \frac{-1}{2} [\operatorname{cosec} x \operatorname{cot} x - \ln |\operatorname{cosec} x - \operatorname{cot} x|] + c \quad \text{Ans.}
 \end{aligned}$$

Q.3 Show that $\int e^{ax} \sin bx dx = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin(bx - \tan^{-1} \frac{b}{a}) + c$ (Guj. Board 2005)

Solution:

Taking

Let

$$\begin{aligned}
 I &= \int e^{ax} \sin bx dx \\
 &\quad \because e^{ax} \rightarrow 1^{\text{st}} \text{ function} \\
 &\quad \sin bx \rightarrow 2^{\text{nd}} \text{ function}
 \end{aligned}$$

$$\begin{aligned}
 I &= e^{ax} \int \sin bx dx - \int [\int \sin bx dx \cdot \frac{d}{dx} (e^{ax})] dx \\
 I &= e^{ax} \left(\frac{-\cos bx}{b} \right) - \int \frac{-\cos bx}{b} \cdot e^{ax} \cdot adx \\
 I &= \frac{-e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx dx \\
 I &= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \left[e^{ax} \int \cos bx dx - \int \left(\int \cos bx dx \cdot \frac{d}{dx} (e^{ax}) \right) dx \right] \\
 I &= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} e^{ax} \cdot \frac{\sin bx}{b} - \frac{a}{b} \int \frac{\sin bx}{b} \cdot ae^{ax} dx \\
 I &= -\frac{e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} - \frac{a^2}{b^2} \int e^{ax} \sin bx dx \\
 I &= -\frac{e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} - \frac{a^2}{b^2} I + c
 \end{aligned}$$

$\because e^{ax} \rightarrow 1^{\text{st}}$ function
 $\cos bx \rightarrow 2^{\text{nd}}$ function

$$\begin{aligned} I + \frac{a^2}{b^2} I &= -\frac{e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} + c \\ \left(\frac{a^2 + b^2}{b^2}\right) I &= -\frac{e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} + c \\ I &= \frac{-b^2 e^{ax} \cos bx}{b(a^2 + b^2)} + \frac{ab^2 e^{ax} \sin bx}{b^2(a^2 + b^2)} + c \\ I &= \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c \end{aligned}$$

Put

$$a = r \cos \theta \quad \text{--- (1)}$$

$$b = r \sin \theta \quad \text{--- (2)}$$

Squaring and adding equation (1) and equation (2)

$$a^2 + b^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$a^2 + b^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$r^2 = a^2 + b^2$$

$$r = \sqrt{a^2 + b^2}$$

Dividing equation (2) by equation (1)

$$\frac{b}{a} = \frac{r \sin \theta}{r \cos \theta}$$

$$\tan \theta = \frac{b}{a}$$

$$\theta = \tan^{-1} \frac{b}{a}$$

$$I = \frac{e^{ax}}{a^2 + b^2} [\sin bx \cdot r \cos \theta - r \sin \theta \cos bx] + c$$

$$I = \frac{re^{ax}}{a^2 + b^2} [\sin bx \cos \theta - \cos bx \sin \theta] + c$$

$$I = \frac{\sqrt{a^2 + b^2} e^{ax}}{a^2 + b^2} [\sin(bx - \theta)] + c$$

$$I = \frac{e^{ax}}{\sqrt{a^2 + b^2}} [\sin(bx - \tan^{-1} \frac{b}{a})] + c \quad \text{Hence proved.}$$

Q.4 Evaluate the following indefinite integrals.

$$(i) \int \sqrt{a^2 - x^2} dx \quad (ii) \int \sqrt{x^2 - a^2} dx \quad (iii) \int \sqrt{4 - 5x^2} dx$$

$$(iv) \int \sqrt{3 - 4x^2} dx \quad (v) \int \sqrt{x^2 + 4} dx \quad (vi) \int x^2 e^{ax} dx$$

Solution:

$$(i) \int \sqrt{a^2 - x^2} dx \quad (\text{Lhr. Board 2007, 2009}) (\text{Guj. Board 2008})$$

$$\text{Put } x = a \sin\theta \Rightarrow \sin\theta = x$$

$$dx = a \cos\theta d\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= \int \sqrt{a^2 - a^2 \sin^2\theta} \cdot a \cos\theta d\theta$$

$$= \int \sqrt{a^2(1 - \sin^2\theta)} \cdot a \cos\theta d\theta$$

$$= \int a \sqrt{\cos^2\theta} \cdot a \cos\theta d\theta$$

$$= a^2 \int \cos\theta \cdot \cos\theta d\theta$$

$$= a^2 \int \cos^2\theta d\theta$$

$$= a^2 \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{a^2}{2} \int (1 + \cos 2\theta) d\theta$$

$$= \frac{a^2}{2} \int d\theta + \frac{a^2}{2} \int \cos 2\theta d\theta$$

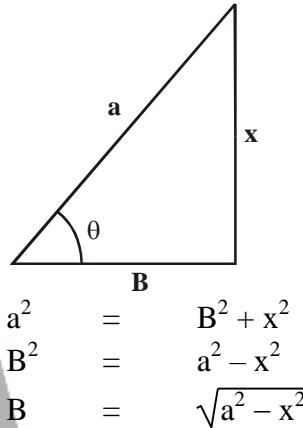
$$= \frac{a^2}{2} \theta + \frac{a^2}{2} \cdot \frac{\sin 2\theta}{2} + C$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{4} 2\sin\theta \cos\theta + C$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2} \cdot \frac{x}{a} \cos\theta + C$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2} \cdot \frac{\sqrt{a^2 - x^2}}{a} + C$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x \sqrt{a^2 - x^2}}{2} + C \quad \text{Ans.}$$



Alternative Solution

$$(i) \int \sqrt{a^2 - x^2} \cdot 1 \cdot dx$$

$$\therefore \sqrt{a^2 - x^2} \rightarrow 1^{\text{st}} \text{ function}$$

$$1 \rightarrow 2^{\text{nd}} \text{ function}$$

Now

$$\begin{aligned}
 &= \sqrt{a^2 - x^2} \int 1 dx - \int \left[\int 1 dx \cdot \frac{d}{dx} \sqrt{a^2 - x^2} \right] dx \\
 &= \sqrt{a^2 - x^2} \cdot x - \int x \cdot \frac{-1(2x)}{2\sqrt{a^2 - x^2}} dx \\
 &= \sqrt{a^2 - x^2} \cdot x - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx \\
 &= x \cdot \sqrt{a^2 - x^2} - \int \frac{-a^2 + a^2 - x^2}{\sqrt{a^2 - x^2}} dx \\
 &= x \sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx - \int \sqrt{a^2 - x^2} dx \\
 &= x \sqrt{a^2 - x^2} + a^2 \cdot \sin^{-1} \left(\frac{x}{a} \right) - I \\
 2I &= x \sqrt{a^2 - x^2} + a^2 \cdot \sin^{-1} \left(\frac{x}{a} \right) + C
 \end{aligned}$$

Dividing both sides by 2

$$I = \frac{x \sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$(ii) \quad \sqrt{x^2 - a^2} \ dx$$

$$\text{Put } x = a \sec \theta \Rightarrow \sec \theta = \frac{x}{a}$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$= \int \sqrt{a^2 \sec^2 \theta - a^2} \cdot a \sec \theta \tan \theta d\theta$$

$$= \int \sqrt{a^2 (\sec^2 \theta - 1)} \cdot a \sec \theta \tan \theta d\theta$$

$$= \int a \sqrt{\tan^2 \theta} \cdot a \sec \theta \tan \theta d\theta$$

$$= a^2 \int \tan \theta \cdot \sec \theta \tan \theta d\theta$$

$$= a^2 \int \tan^2 \theta \sec \theta d\theta$$

$$= a^2 \int (\sec^2 \theta - 1) \sec \theta d\theta$$

$$\begin{aligned}
 &= a^2 \int \sec^3 \theta d\theta - a^2 \int \sec \theta d\theta \\
 &= a^2 I - a^2 \ln |\sec \theta + \tan \theta| + c_1 \quad \text{---} \quad (1)
 \end{aligned}$$

Where

$$I = \int \sec^3 \theta d\theta$$

$$I = \int \sec \theta \cdot \sec^2 \theta d\theta$$

$$I = \sec \theta \int \sec^2 \theta d\theta - \int [\int \sec^2 \theta d\theta \cdot \frac{d}{d\theta} (\sec \theta)] d\theta$$

$$I = \sec \theta \tan \theta - \int \tan \theta \cdot \sec \theta \tan \theta d\theta$$

$$I = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta$$

$$I = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta$$

$$I = \sec \theta \tan \theta - \sec^3 \theta d\theta + \int \sec \theta d\theta$$

$$I = \sec \theta \tan \theta - I + \ln |\sec \theta + \tan \theta| + c_2$$

$$2I = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| + c_2$$

$$I = \frac{\sec \theta \tan \theta}{2} + \frac{1}{2} \ln |\sec \theta + \tan \theta| + c_2$$

∴ From equation (1)

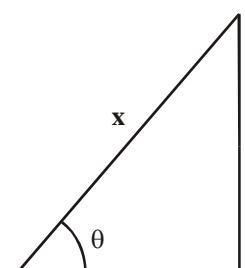
$$= \frac{a^2 \sec \theta \tan \theta}{2} + \frac{a^2}{2} \ln |\sec \theta + \tan \theta| - a^2 \ln |\sec \theta + \tan \theta| + c_1 + c_2$$

$$= \frac{a^2 \sec \theta \tan \theta}{2} + \left(\frac{a^2}{2} - a^2 \right) \ln |\sec \theta + \tan \theta| + c_1 + c_2$$

$$= \frac{a^2}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \frac{a^2}{2} \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + c_1 + c_2$$

$$= \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + c_1 + c_2$$

$$= \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + \frac{a^2}{2} \ln a + c_1 + c_2$$



$$\begin{aligned}
 x^2 &= a^2 + P^2 \\
 P^2 &= x^2 - a^2 \\
 P &= \sqrt{x^2 - a^2}
 \end{aligned}$$

$$= \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + c \text{ Ans. where } c = \frac{a^2}{2} \ln a + c_1 + c_2$$

Alternative Method

$$(ii) \int \sqrt{x^2 - a^2} dx$$

$\therefore \sqrt{x^2 - a^2} \rightarrow 1^{\text{st}}$ function

$1 \rightarrow 2^{\text{nd}}$ function

Let

$$\begin{aligned} I &= \int \sqrt{x^2 - a^2} \cdot 1 \cdot dx \\ &= \sqrt{x^2 - a^2} \cdot x - \int \frac{1}{2\sqrt{x^2 - a^2}} (2x) dx \\ &= x \sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx \\ &= x \sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx \\ &= x \sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}} \end{aligned}$$

$$2I = x \sqrt{x^2 - a^2} - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}} \dots\dots\dots (1)$$

We know that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + c$$

Put it in (1)

$$I = \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + c$$

$$(iii) \int \sqrt{4 - 5x^2} dx$$

$$= \int \sqrt{5 \left(\frac{4}{5} - x^2 \right)} dx$$

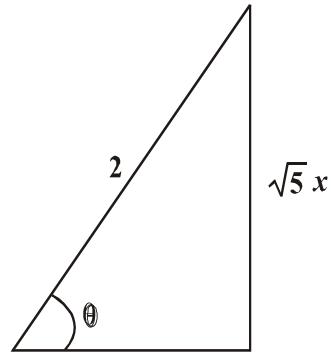
$$= \sqrt{5} \int \sqrt{\left(\frac{2}{\sqrt{5}} \right)^2 - x^2} dx$$

Put

$$x = \frac{2}{\sqrt{5}} \sin \theta \Rightarrow \sin \theta = \frac{\sqrt{5}x}{2}$$

$$dx = \frac{2}{\sqrt{5}} \cos \theta d\theta \Rightarrow \theta = \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right)$$

$$\begin{aligned}
 &= \sqrt{5} \int \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}} \sin \theta\right)^2} \cdot \frac{2}{\sqrt{5}} \cos \theta d\theta \\
 &= \frac{2\sqrt{5}}{\sqrt{5}} \int \sqrt{\frac{4}{5} - \frac{4}{5} \sin^2 \theta} \cdot \cos \theta d\theta \\
 &= 2 \int \sqrt{\frac{4}{5}(1 - \sin^2 \theta)} \cos \theta d\theta \\
 &= 2 \left(\frac{2}{\sqrt{5}}\right) \int \sqrt{\cos^2 \theta} \cos \theta d\theta \\
 &= \frac{4}{\sqrt{5}} \int \cos \theta \cos \theta d\theta \\
 &= \frac{4}{\sqrt{5}} \int \cos^2 \theta d\theta \\
 &= \frac{4}{\sqrt{5}} \int \frac{1 + \cos 2\theta}{2} d\theta \\
 &= \frac{2}{\sqrt{5}} \int d\theta + \frac{2}{\sqrt{5}} \int \cos 2\theta d\theta \\
 &= \frac{2}{\sqrt{5}} \theta + \frac{2}{\sqrt{5}} \cdot \frac{\sin 2\theta}{2} + c \\
 &= \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + \frac{1}{\sqrt{5}} \cdot 2 \sin \theta \cos \theta + c \\
 &= \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}x}{2} \cdot \frac{\sqrt{4-5x^2}}{2} + c \\
 &= \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + \frac{x}{2} \sqrt{4-5x^2} + c \text{ Ans.}
 \end{aligned}$$



$$\begin{aligned}
 4 &= B^2 + 5x^2 \\
 B^2 &= 4 - 5x^2 \\
 B &= \sqrt{4 - 5x^2}
 \end{aligned}$$

Alternative Method

$$\begin{aligned}
 I &= \int \sqrt{4 - 5x^2} \cdot 1 \cdot dx \\
 \therefore \sqrt{4 - 5x^2} &\rightarrow 1^{\text{st}} \text{ function} \\
 1 \cdot &\rightarrow 2^{\text{nd}} \text{ function} \\
 &= \sqrt{4 - 5x^2} \int 1 dx - \left[\int i dx \frac{d}{dx} \sqrt{4 - 5x^2} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{4 - 5x^2} \cdot x - \int x \cdot \frac{1}{2\sqrt{4 - 5x^2}} (-10x) dx \\
 &= x \sqrt{4 - 5x^2} - \int \frac{-5x^2}{\sqrt{4 - 5x^2}} dx \\
 &= x \sqrt{4 - 5x^2} - \int \frac{-4 + 4 - 5x^2}{\sqrt{4 - 5x^2}} dx \\
 &= x \sqrt{4 - 5x^2} + 4 \int \frac{1}{\sqrt{4 - 5x^2}} dx - \int \sqrt{4 - 5x^2} dx \\
 &= x \sqrt{4 - 5x^2} + 4 \int \frac{1}{\sqrt{4 - 5x^2}} dx - I \\
 2I &= x \sqrt{4 - 5x^2} + \frac{4}{\sqrt{5}} \cdot \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + c
 \end{aligned}$$

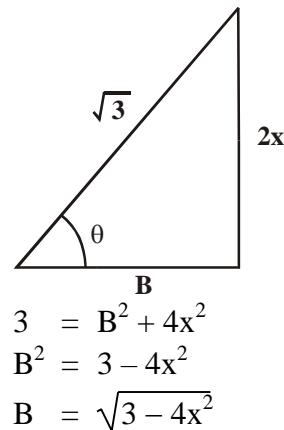
÷ by 2 we know that

$$\begin{aligned}
 \int \frac{1}{\sqrt{4 - 5x^2}} dx &= \frac{1}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + c \\
 I &= \frac{x\sqrt{4 - 5x^2}}{2} + \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + c \\
 \text{(iv)} \quad \int \sqrt{3 - 4x^2} dx &\quad (\text{Guj. Board 2005}) \\
 &= \int \sqrt{4 \left(\frac{3}{4} - x^2 \right)} dx \\
 &= 2 \int \sqrt{\left(\frac{\sqrt{3}}{2} \right)^2 - x^2} dx
 \end{aligned}$$

Put

$$\begin{aligned}
 x &= \frac{\sqrt{3}}{2} \sin \theta \Rightarrow \sin \theta = \frac{2x}{\sqrt{3}} \\
 dx &= \frac{\sqrt{3}}{2} \cos \theta d\theta \Rightarrow \theta = \sin^{-1} \left(\frac{2x}{\sqrt{3}} \right) \\
 &= 2 \sqrt{\left(\frac{\sqrt{3}}{2} \right)^2 - \left(\frac{\sqrt{3}}{2} \sin \theta \right)^2} \cdot \frac{\sqrt{3}}{2} \cos \theta d\theta \\
 &= \frac{2\sqrt{3}}{2} \int \sqrt{\frac{3}{4} - \frac{3}{4} \sin^2 \theta} \cdot \cos \theta d\theta \\
 &= \sqrt{3} \int \sqrt{\frac{3}{4} (1 - \sin^2 \theta)} \cos \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{3} \left(\frac{\sqrt{3}}{2} \right) \int \sqrt{\cos^2 \theta} \cdot \cos \theta d\theta \\
 &= \frac{3}{2} \int \cos \theta \cdot \cos \theta d\theta \\
 &= \frac{3}{2} \int \cos^2 \theta d\theta \\
 &= \frac{3}{2} \int \frac{1 + \cos 2\theta}{2} d\theta \\
 &= \frac{3}{4} \int d\theta + \frac{3}{4} \int \cos 2\theta d\theta \\
 &= \frac{3}{4} \theta + \frac{3}{4} \cdot \frac{\sin 2\theta}{2} + c \\
 &= \frac{3}{4} \theta + \frac{3}{8} \cdot 2 \sin \theta \cos \theta + c \\
 &= \frac{3}{4} \sin^{-1} \left(\frac{2x}{\sqrt{3}} \right) + \frac{3}{4} \cdot \frac{2x}{\sqrt{3}} \cdot \frac{\sqrt{3 - 4x^2}}{\sqrt{3}} + c \\
 &= \frac{3}{4} \sin^{-1} \left(\frac{2x}{\sqrt{3}} \right) + \frac{x}{2} \sqrt{3 - 4x^2} + c
 \end{aligned}$$



Alternative Method

$$I = \int \sqrt{3 - 4x^2} dx$$

Let

$$\begin{aligned}
 \sqrt{3 - 4x^2} &\rightarrow 1^{\text{st}} \text{ function} \\
 1 &\rightarrow 2^{\text{nd}} \text{ function} \\
 &= \sqrt{3 - 4x^2} \cdot \int 1 dx - \int \left[\int 1 dx \cdot \frac{d}{dx} \sqrt{3 - 4x^2} \right] dx \\
 &= \sqrt{3 - 4x^2} \cdot x - \int x \cdot \frac{1}{2\sqrt{3 - 4x^2}} (-8x) dx \\
 &= \sqrt{3 - 4x^2} \cdot x - \int \frac{-4x^2}{\sqrt{3 - 4x^2}} dx \\
 &= \sqrt{3 - 4x^2} \cdot x - \int \frac{-3 + 3 - 4x^2}{\sqrt{3 - 4x^2}} dx \\
 &= x \cdot \sqrt{3 - 4x^2} + 3 \int \frac{1}{\sqrt{3 - 4x^2}} dx - \int \sqrt{3 - 4x^2} dx
 \end{aligned}$$

$$\begin{aligned}
 &= x \cdot \sqrt{3 - 4x^2} + 3 \int \frac{1}{\sqrt{3 - 4x^2}} dx - I \\
 I + I &= x \sqrt{3 - 4x^2} + 3 \int \frac{1}{\sqrt{3 - 4x^2}} dx
 \end{aligned}$$

We know that

$$\begin{aligned}
 \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \sin^{-1}\left(\frac{x}{a}\right) \\
 &= x \sqrt{3 - 4x^2} + \frac{3}{2} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c \\
 I &= \frac{x \sqrt{3 - 4x^2}}{2} + \frac{3}{4} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 (v) \quad &\int \sqrt{x^2 + 4} dx \\
 &\int \sqrt{x^2 + (2)^2} dx
 \end{aligned}$$

Put

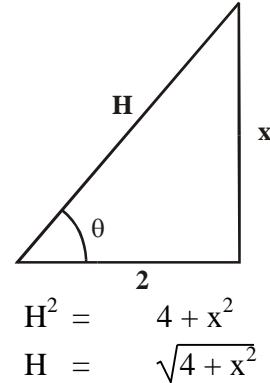
$$\begin{aligned}
 x &= 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2} \\
 dx &= 2\sec^2\theta d\theta \\
 &= \int \sqrt{4\tan^2\theta + 4} \cdot 2\sec^2\theta d\theta \\
 &= \int \sqrt{4(\tan^2\theta + 1)} \cdot 2\sec^2\theta d\theta \\
 &= \int 2\sqrt{\sec^2\theta} \cdot 2\sec^2\theta d\theta \\
 &= 4 \int \sec\theta \cdot \sec^2\theta d\theta \\
 &= 4 \int \sec^3\theta d\theta \\
 &= 4I \quad \text{---} \quad (1) \\
 I &= \int \sec^3\theta d\theta \\
 &= \int \sec\theta \cdot \sec^2\theta d\theta \\
 &= \sec\theta \int \sec^2\theta d\theta - \int \left[\int \sec^2\theta d\theta \cdot \frac{d}{d\theta}(\sec\theta) \right] d\theta \\
 &= \sec\theta \cdot \tan\theta - \int \tan\theta \cdot \sec\theta \tan\theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 &= \sec\theta \tan\theta - \int \sec\theta \tan^2\theta d\theta \\
 &= \sec\theta \tan\theta - \int \sec\theta (\sec^2\theta - 1) d\theta \\
 &= \sec\theta \tan\theta - \int \sec^3\theta d\theta + \sec\theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 I &= \sec\theta \tan\theta - I + \ln |\sec\theta + \tan\theta| + c_1 \\
 2I &= \sec\theta \tan\theta + \ln |\sec\theta + \tan\theta| + c_1
 \end{aligned}$$

∴ From equation (1)

$$\begin{aligned}
 &= 4I \\
 &= 2(2I) \\
 &= 2[\sec\theta \tan\theta + \ln |\sec\theta + \tan\theta|] + c_1 \\
 &= 2 \frac{\sqrt{4+x^2}}{2} \cdot \frac{x}{2} + 2 \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + c_1 \\
 &= \frac{x\sqrt{4+x^2}}{2} + 2 \ln \left| \frac{\sqrt{4+x^2}+x}{2} \right| + c_1 \\
 &= \frac{x\sqrt{4+x^2}}{2} + 2 \ln |\sqrt{4+x^2}+x| - 2\ln 2 + c_1 \\
 &= \frac{x\sqrt{4+x^2}}{2} + 2 \ln |\sqrt{4+x^2}+x| + c \quad \text{where } c = -2\ln 2 + c_1
 \end{aligned}$$



Alternative Method

$$I = \int \sqrt{x^2 + 4} dx$$

Let

$$\sqrt{x^2 + 4} \rightarrow 1^{\text{st}} \text{ function}$$

$$1. \rightarrow 2^{\text{nd}} \text{ function}$$

$$\begin{aligned}
 I &= \sqrt{x^2 + 4} \cdot \int 1 dx - \int \left[\int 1 dx \cdot \frac{d}{dx} \sqrt{x^2 + 4} \right] dx \\
 &= x \cdot \sqrt{x^2 + 4} - \int x \cdot \frac{1}{2\sqrt{x^2 + 4}} (2x) dx \\
 &= x \cdot \sqrt{x^2 + 4} - \int \frac{x^2}{\sqrt{x^2 + 4}} dx \\
 &= x \cdot \sqrt{x^2 + 4} - \int \frac{-4 + x^2 + 4}{\sqrt{x^2 + 4}} dx
 \end{aligned}$$

$$= x \cdot \sqrt{x^2 + 4} + 4 \int \frac{1}{\sqrt{x^2 + 4}} dx - \int \sqrt{x^2 + 4} dx$$



$$\begin{aligned}
 &= x \cdot \sqrt{x^2 + 4} + 4 \int \frac{1}{\sqrt{x^2 + 4}} dx - I \\
 2I &= x \cdot \sqrt{x^2 + 4} + 4 \int \frac{1}{\sqrt{x^2 + 4}} dx \\
 I &= \frac{x\sqrt{x^2 + 4}}{2} + 2 \int \frac{1}{\sqrt{x^2 + 4}} dx \\
 I &= \frac{x\sqrt{x^2 + 4}}{2} + 2 \ln |x + \sqrt{x^2 + 4}| + C \\
 \therefore \int \frac{1}{\sqrt{x^2 + a^2}} dx &= \ln |x + \sqrt{x^2 + a^2}| + C \quad \text{Ans.}
 \end{aligned}$$

(vi) $\int x^2 e^{ax} dx$

$$\begin{aligned}
 &= x^2 \int e^{ax} dx - \int \left[\int e^{ax} dx \cdot \frac{d}{dx}(x^2) \right] dx \\
 &= x^2 \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot 2x dx \\
 &= \frac{x^2 e^{ax}}{a} - \frac{2}{a} \int x e^{ax} dx \\
 &= \frac{x^2 e^{ax}}{a} - \frac{2}{a} \left[x \int e^{ax} dx - \int \left(\int e^{ax} dx \cdot \frac{d}{dx}(x) \right) dx \right] \\
 &= \frac{x^2 e^{ax}}{a} - \frac{2}{a} x \frac{e^{ax}}{a} + \frac{2}{a} \int \frac{e^{ax}}{a} \cdot 1 \cdot dx \\
 &= \frac{x^2 e^{ax}}{a} - \frac{2x e^{ax}}{a^2} + \frac{2}{a^2} \int e^{ax} dx \\
 &= \frac{x^2 e^{ax}}{a} - \frac{2x e^{ax}}{a^2} + \frac{2}{a^2} \frac{e^{ax}}{a} + C \\
 &= \frac{e^{ax}}{a} \left(x^2 - \frac{2x}{a} + \frac{2}{a^2} \right) + C \quad \text{Ans.}
 \end{aligned}$$

Q.5 Evaluate the following integrals.

(i) $\int e^x \left(\frac{1}{x} + \ln x \right) dx$

(ii) $\int e^x (\cos x + \sin x) dx$

(iii) $\int e^{ax} \left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2 - 1}} \right] dx$

(iv) $\int e^{3x} \left(\frac{3\sin x - \cos x}{\sin^2 x} \right) dx$

(v) $\int e^{2x} [-\sin x + 2\cos x] dx$

(vi) $\int \frac{xex}{(1+x)^2} dx$

(vii) $\int e^{-x} (\cos x - \sin x) dx$ (L.B 2009) (S) (viii) $\int \frac{e^{mtan^{-1}x}}{1+x^2} dx$

(ix) $\int \frac{2x}{1-\sin x} dx$

(x) $\int \frac{e^x (1+x)}{(2+x)^2} dx$

(xi) $\int \left(\frac{1-\sin x}{1-\cos x} \right) e^x dx$

Solution:

(i) $\int e^x \left(\frac{1}{x} + \ln x \right) dx$

= $\int e^x \left(\ln x + \frac{1}{x} \right) dx$

$$\because f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

By using the formula

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f'(x) + c$$

= $e^x \ln x + c$ Ans.

(ii) $\int e^x (\cos x + \sin x) dx$

By using the formula

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

= $e^x \sin x + c$ Ans.

$$\because f(x) = \sin x$$

$$f'(x) = \cos x$$

(iii) $\int e^{ax} \left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2 - 1}} \right] dx$ (Lhr. Board 2009)

By using the formula

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

= $e^{ax} \sec^{-1} x + c$ Ans.

$$\because f(x) = \sec^{-1} x$$

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\begin{aligned}
 \text{(iv)} \quad & \int e^{3x} \left(\frac{3 \sin x - \cos x}{\sin^2 x} \right) dx \\
 &= \int e^{3x} \left(\frac{3 \sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx \\
 &= \int e^{3x} \left(\frac{3}{\sin x} - \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \right) dx \\
 &= \int e^{3x} (3 \operatorname{cosec} x - \operatorname{cosec} x \cot x) dx
 \end{aligned}$$

By using the formula

$$\begin{aligned}
 \because f(x) &= \operatorname{cosec} x \\
 f'(x) &= -\cos x \cot x
 \end{aligned}$$

$$\begin{aligned}
 \int e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= e^{3x} \operatorname{cosec} x + c \quad \text{Ans.}
 \end{aligned}$$

$$\text{(v)} \quad \int e^{2x} [-\sin x + 2 \cos x] dx$$

By using the formula

$$\begin{aligned}
 \because f(x) &= \cos x \\
 f'(x) &= -\sin x
 \end{aligned}$$

$$\text{(vi)} \quad \int \frac{x e^x}{(1+x)^2} dx \quad (\text{Guj. Board 2007})$$

$$\begin{aligned}
 &= \int \frac{(1+x-1)e^x}{(1+x)^2} dx \\
 &= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx \\
 &= \int e^x \left[\frac{1}{1+x} - \frac{1}{(1+x)^2} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 \because f(x) &= \frac{1}{1+x} \\
 f'(x) &= -\frac{1}{(1+x)^2}
 \end{aligned}$$

By using the formula

$$\begin{aligned}
 \int e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= \frac{e^x}{1+x} + c \quad \text{Ans.}
 \end{aligned}$$

(vii)

$$\int e^{-x} (\cos x - \sin x) dx$$

By using the formula

$$\begin{aligned}
 e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= e^{-x} \sin x + c
 \end{aligned}$$

$$\int e^{-x} (-\sin x + \cos x) dx$$

$$f(x) = \sin x$$

$$f(x) = \cos x$$

$$\begin{aligned}
 \text{(viii)} \quad & \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx \\
 &= \frac{1}{m} \int e^{m \tan^{-1} x} \frac{m}{1+x^2} \\
 &= \frac{1}{m} e^{m \tan^{-1} x} + c \quad \text{Ans.}
 \end{aligned}$$

Alternative Method

$$\text{Let } t = \tan^{-1} x$$

$$\begin{aligned}
 dt &= \frac{1}{1+x^2} dx \\
 \therefore \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx &= \int e^{mt} dx \\
 &= \frac{e^{mt}}{m} + c \\
 &= \frac{1}{m} e^{m \tan^{-1} x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ix)} \quad & \int \frac{2x}{1-\sin x} dx \\
 & \int \frac{2x}{1-\cos(\frac{\pi}{2}-x)} dx \\
 &= \int \frac{2x}{2\sin^2\left(\frac{\pi}{2}-\frac{x}{2}\right)} dx \\
 &= \int x \operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right) dx
 \end{aligned}$$



$$\begin{aligned}
 &= x \int \operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right) dx - \int \left[\int \operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right) dx \cdot \frac{d}{dx}(x) \right] dx \\
 &= x \cdot \frac{-\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\frac{-1}{2}} - \int \frac{-\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\frac{-1}{2}} \cdot 1 dx \\
 &= 2x \cot\left(\frac{\pi}{4}-\frac{x}{2}\right) - 2 \int \cot\left(\frac{\pi}{4}-\frac{x}{2}\right) dx
 \end{aligned}$$

$$\left. \begin{aligned}
 &\because \cos 2x = 1 - 2\sin^2 x \\
 &\cos x = 1 - 2\sin^2 \frac{x}{2} \\
 &2 \sin^2 \frac{x}{2} = 1 - \cos x
 \end{aligned} \right\}$$

$\therefore x \rightarrow 1^{\text{st}}$ function
 $\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right) \rightarrow 2^{\text{nd}}$
 function

$$\begin{aligned}
 &= 2x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) - \frac{2 \ln \left| \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) \right|}{\frac{-1}{2}} + c \\
 &= 2x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) + 4 \ln \left| \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) \right| + c \quad \text{Ans.}
 \end{aligned}$$

(xi) $\int \frac{e^x(1+x)}{(2+x)^2} dx$

$$\begin{aligned}
 &= \int \frac{e^x(1+x+1-1)}{(2+x)^2} dx \\
 &= \int \frac{e^x(2+x-1)}{(2+x)^2} dx \\
 &= \int e^x \left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right] dx \\
 &= \int e^x \left[\frac{1}{2+x} - \frac{1}{(2+x)^2} \right] dx
 \end{aligned}$$

$$\therefore f(x) = \frac{1}{2+x}$$

$$f'(x) = \frac{-1}{(2+x)^2}$$

By using the formula

$$\begin{aligned}
 \int e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= \frac{e^x}{2+x} + c \quad \text{Ans.}
 \end{aligned}$$

(xi) $\int \left(\frac{1-\sin x}{1-\cos x} \right) e^x dx$

$$\begin{aligned}
 &= \int e^x \left(\frac{1 - 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) dx \\
 &= \int e^x \left(\frac{1}{2 \sin^2 \frac{x}{2}} - \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx \\
 &= \int \left(\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \right) e^x dx \\
 &= - \int \left(\frac{-1}{2} \operatorname{cosec}^2 \frac{x}{2} + \cot \frac{x}{2} \right) e^x dx
 \end{aligned}$$

$$\therefore f(x) = \cot \frac{x}{2}$$

$$f'(x) = -\operatorname{csc}^2 \frac{x}{2} \left(\frac{1}{2} \right)$$

$$f'(x) = -\frac{1}{2} \operatorname{csc}^2 \left(\frac{x}{2} \right)$$