

Integrate

$$\begin{aligned}
 & \int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} dx = \int \frac{2x + 1 + 2}{x^2 + x + 1} dx + \frac{1}{2} \int \frac{2x - 8 + 2 - 2}{x^2 + 2x + 3} dx \\
 &= \int \frac{2x + 1}{x^2 + x + 1} dx + 2 \int \frac{dx}{x^2 + x + 1} + \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 3} dx - \frac{10}{2} \int \frac{dx}{x^2 + 2x + 3} \\
 &= \ln|x^2+x+1| + 2 \int \frac{dx}{x^2 + x + \frac{1}{4} - \frac{1}{4} + 1} + \frac{1}{2} \ln|x^2+2x+3| - 5 \int \frac{dx}{x^2 + 2x + 1 - 1 + 3} \\
 &= \ln|x^2 + x + 1| + 2 \int \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}} + \ln|x^2 + 2x + 3|^{\frac{1}{2}} - 5 \int \frac{dx}{(x + 1)^2 + 2} \\
 &= \ln|(x^2 + x + 1)\sqrt{x^2 + 2x + 3}| + 2 \int \frac{dx}{(x + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - 5 \int \frac{dx}{(x + 1)^2 + (\sqrt{2})^2} \\
 &= \ln|(x^2 + x + 1)\sqrt{x^2 + 2x + 3}| + 2 \cdot \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x + \frac{1}{2}}{\sqrt{3}/2}\right) - 5 \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x + 1}{\sqrt{2}}\right) + C \\
 &= \ln|(x^2 + x + 1)\sqrt{x^2 + 2x + 3}| + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) - \frac{5}{\sqrt{2}} \tan^{-1}\left(\frac{x + 1}{\sqrt{2}}\right) + C
 \end{aligned}$$

EXERCISE 3.6

Q.1 $\int_1^2 (x^2 + 1) dx$

Solution:

$$\begin{aligned}
 & \int_1^2 (x^2 + 1) dx \\
 &= \int_1^2 x^2 dx + \int_1^2 1 dx \\
 &= \left[\frac{x^3}{3} \right]_1^2 + [x]_1^2
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_a^b f(x) dx &= [\varphi(x)]_a^b \\
 &= \varphi(b) - \varphi(a)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{3} (2^3 - 1^3) + (2 - 1) \\
 &= \frac{1}{3} (8 - 1) + 1 = \frac{7}{3} + 1 \\
 &= \frac{7+3}{3} = \frac{10}{3} \quad \text{Ans.}
 \end{aligned}$$

Q.2 $\int_{-1}^1 (x^{1/3} + 1) dx$

Solution:

$$\begin{aligned}
 &\int_{-1}^1 (x^{1/3} + 1) dx \\
 &= \int_{-1}^1 x^{1/3} dx + \int_{-1}^1 1 dx \\
 &= \left[\frac{x^{2/3}}{\frac{2}{3}} \right]_{-1}^1 + [x]_{-1}^1 \\
 &= \frac{3}{2} [(1)^{2/3} - (-1)^{2/3}] + [1 - (-1)] \\
 &= \frac{3}{2} (1 - 1) + 2 \\
 &= \frac{3}{2} (0) + 2 = 2 \quad \text{Ans.}
 \end{aligned}$$

Q.3 $\int_{-2}^0 \frac{1}{(2x-1)^2} dx$

Solution:

$$\begin{aligned}
 &\int_{-2}^0 \frac{1}{(2x-1)^2} dx \\
 &= \frac{1}{2} \int_{-2}^0 (2x-1)^{-2} \cdot 2 dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\frac{(2x - 1)^{-2+1}}{-2 + 1} \right]_0^{-2} \\
 &= \frac{1}{2} \left[\frac{(2x - 1)^{-1}}{-1} \right]_0^{-2} \\
 &= \frac{-1}{2} \left[\frac{1}{2x - 1} \right]_0^{-2} \\
 &= \frac{-1}{2} \left[\frac{1}{2(0) - 1} - \frac{1}{2(-2) - 1} \right] \\
 &= \frac{-1}{2} \left[\frac{1}{0 - 1} - \frac{1}{-4 - 1} \right] \\
 &= \frac{-1}{2} (-1 + \frac{1}{5}) \\
 &= \frac{-1}{2} \left(\frac{-5 + 1}{5} \right) \\
 &= \frac{-1(-4)}{10} \\
 &= \frac{2}{5} \quad \text{Ans.}
 \end{aligned}$$

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Q.4 $\int_{-6}^2 \sqrt{3-x} \, dx$

Solution:

$$\begin{aligned}
 &\int_{-6}^2 \sqrt{3-x} \, dx \\
 &= -\int_{-6}^2 (3-x)^{1/2} \, -dx \\
 &= -\left[\frac{(3-x)^{2/3}}{\frac{3}{2}} \right]_{-6}^2 \\
 &= \frac{-2}{3} [(3-2)^{2/3} - (3+6)^{2/3}] \\
 &= \frac{-2}{3} [(1)^{2/3} - (9)^{2/3}]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{-2}{3} [1 - (3^2)^{2/3}] \\
 &= \frac{-2}{3} (1 - 27) \\
 &= \frac{-2}{3} (-26) \\
 &= \frac{52}{3} \quad \text{Ans.}
 \end{aligned}$$

Q.5 $\int_1^{\sqrt{5}} \sqrt{(2t-1)^3} dt$

Solution:

$$\begin{aligned}
 A &= \int_1^{\sqrt{5}} \sqrt{(2t-1)^3} dt \\
 &= \frac{1}{2} \int_1^{\sqrt{5}} (2t-1)^3 \cdot 2dt \\
 &= \frac{1}{2} \left[\frac{(2t-1)^{5/2}}{\frac{5}{2}} \right]_1^{\sqrt{5}} \\
 &= \frac{1}{5} [(2\sqrt{5}-1)^{5/2} - (2-1)^{5/2}] \\
 &= \frac{1}{5} [(2\sqrt{5}-1)^{5/2} - 1] \quad \text{Ans.}
 \end{aligned}$$

Q.6 $\int_2^{\sqrt{5}} x\sqrt{x^2-1} dx$

Solution:

$$\begin{aligned}
 &\int_2^{\sqrt{5}} x\sqrt{x^2-1} dx \\
 &= \frac{1}{2} \int_2^{\sqrt{5}} (x^2-1)^{1/2} \cdot 2x dx \\
 &= \frac{1}{2} \left[\frac{(x^2-1)^{3/2}}{3/2} \right]_2^{\sqrt{5}}
 \end{aligned}$$



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$$= \frac{1}{3} [(5-1)^{3/2} - (4-1)^{3/2}]$$



$$= \frac{1}{3} [4^{3/2} - 3^{3/2}] = \frac{1}{3} [(2^2)^{3/2} - 3\sqrt{3}]$$

$$= \frac{1}{3} (8 - 3\sqrt{3}) \quad \text{Ans.}$$

Q.7 $\int_1^2 \frac{x}{x^2 + 2} dx$ (Lhr. Board 2011, Guj. Board 2008)

Solution:

$$\int_1^2 \frac{x}{x^2 + 2} dx$$

$$= \frac{1}{2} \int_1^2 \frac{2x}{x^2 + 2} dx$$

$$= \frac{1}{2} [\ln(x^2 + 2)]_1^2$$

$$= \frac{1}{2} \ln(4 + 2) - \ln(1 + 2)$$

$$= \frac{1}{2} (\ln 6 - \ln 3)$$

$$= \frac{1}{2} \ln \frac{6}{3}$$

$$= \frac{1}{2} \ln 2 \quad \text{Ans.}$$

Q.8 $\int_2^3 (x - \frac{1}{x})^2 dx$



Solution:

$$\int_2^3 (x - \frac{1}{x})^2 dx$$

$$= \int_2^3 (x^2 - 2 + \frac{1}{x^2}) dx$$

$$= \int_2^3 x^2 dx - 2 \int_2^3 dx + \int_2^3 x^{-2} dx$$

$$= \left[\frac{x^3}{3} \right]_2^3 - 2 [x]_2^3 + \left[\frac{x-1}{-1} \right]_2^3$$

$$\begin{aligned}
 &= \frac{1}{3}(3^3 - 2^3) - 2(3 - 2) - \left[\frac{1}{x} \right]_2^3 \\
 &= \frac{1}{3}(27 - 8) - 2(1) - \left(\frac{1}{3} - \frac{1}{2} \right) \\
 &= \frac{19}{3} - 2 - \frac{1}{3} + \frac{1}{2} \\
 &= \frac{38 - 12 - 2 + 3}{6} \\
 &= \frac{27}{6} = \frac{9}{2} \quad \text{Ans.}
 \end{aligned}$$

Q.9 $\int_{-1}^1 (x + \frac{1}{2}) \sqrt{x^2 + x + 1} \, dx$ (Guj. Board 2007)

Solution:

$$\begin{aligned}
 &\int_{-1}^1 (x + \frac{1}{2}) \sqrt{x^2 + x + 1} \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (x^2 + x + 1)^{1/2} \cdot 2(x + \frac{1}{2}) \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (x^2 + x + 1)^{1/2} \cdot (2x + 1) \, dx \\
 &= \frac{1}{2} \left[\frac{(x^3 + x + 1)^{3/2}}{\frac{3}{2}} \right]_{-1}^1 \\
 &= \frac{1}{3} \left[(1 + 1 + 1)^{3/2} - (1 - 1 + 1)^{3/2} \right] \\
 &= \frac{1}{3} [3^{3/2} - 1^{3/2}] \\
 &= \frac{1}{3} (3\sqrt{3} - 1) \\
 &= \frac{3\sqrt{3}}{3} - \frac{1}{3} \\
 &= \sqrt{3} - \frac{1}{3} \quad \text{Ans.}
 \end{aligned}$$

Q.10 $\int_0^3 \frac{dx}{x^2 + 9}$

Solution:

$$\begin{aligned}
 & \int_0^3 \frac{dx}{x^2 + 9} \\
 &= \int_0^3 \frac{dx}{(x)^2 + (3)^2} \\
 &= \left[\frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) \right]_0^3 \\
 &= \frac{1}{3} \left[\tan^{-1} \left(\frac{3}{3} \right) - \tan^{-1} \left(\frac{0}{3} \right) \right] \\
 &= \frac{1}{3} [\tan^{-1}(1) - \tan^{-1}(0)] \\
 &= \frac{1}{3} \left(\frac{\pi}{4} - 0 \right) \\
 &= \frac{\pi}{12} \quad \text{Ans.}
 \end{aligned}$$

Q.11 $\int_{\pi/6}^{\pi/3} \cos t dt$



Solution:

$$\begin{aligned}
 \int_0^{\pi/3} \cos t dt &= [\sin t]_{\pi/6}^{\pi/3} \\
 &= \sin \frac{\pi}{3} - \sin \frac{\pi}{6} \\
 &= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2} \quad \text{Ans.}
 \end{aligned}$$

Q.12 $\int_2^1 (x + \frac{1}{x})^{1/2} (1 - \frac{1}{x^2}) dx$ (Lhr. Board 2011)

Solution:

$$\int_2^1 (x + \frac{1}{x})^{1/2} (1 - \frac{1}{x^2}) dx$$

$$\begin{aligned}
 &= \left[\frac{(x + \frac{1}{x})^{3/2}}{\frac{3}{2}} \right]_1^2 \\
 &= \frac{2}{3} \left[(2 + \frac{1}{2})^{3/2} - (1 + \frac{1}{1})^{3/2} \right] \\
 &= \frac{2}{3} \left[\left(\frac{5}{2} \right)^{3/2} - 2^{3/2} \right] \\
 &= \frac{2}{3} \left[\left(\frac{5}{2} \right)^{3/2} - 2\sqrt{2} \right] \\
 &= \frac{2}{3} \left[\frac{5\sqrt{5} - 8}{2\sqrt{2}} \right] = \frac{2}{3} \left[\frac{5\sqrt{5} - 8}{2\sqrt{2}} \right] \\
 &= \frac{1}{3\sqrt{2}} (5\sqrt{5} - 8) \quad \text{Ans.}
 \end{aligned}$$

Q.13 $\int_1^2 \ln x \, dx$



Solution:

$$\begin{aligned}
 &\int_1^2 \ln x \, dx \\
 &= \int_1^2 1 \cdot \ln x \, dx \\
 &= [\ln x \cdot x]_1^2 - \int_1^2 x \cdot \frac{1}{x} \, dx \\
 &= [2 \ln 2 - 1 \ln 1] - \int_1^2 dx \\
 &= (2 \ln 2 - 0) - [x]_1^2 \\
 &= 2 \ln 2 - (2 - 1) \\
 &= 2 \ln 2 - 1 \quad \text{Ans.}
 \end{aligned}$$

Q.14 $\int_0^2 (e^{x/2} - e^{-x/2}) \, dx$

Solution:

$$\int_0^2 (e^{x/2} - e^{-x/2}) \, dx$$

$$\begin{aligned}
 &= \int_0^2 e^{x/2} dx - \int_0^2 e^{-x/2} dx \\
 &= \left[\frac{e^{x/2}}{\frac{1}{2}} \right]_0^2 - \left[\frac{e^{-x/2}}{-\frac{1}{2}} \right]_0^2 \\
 &= 2(e^{2/2} - e^{0/2}) + 2(e^{-2/2} - e^{-0/2}) \\
 &= 2(e - 1 + e^{-1} - 1) \\
 &= 2(e + \frac{1}{e} - 2) \\
 &= 2\left(\frac{e^2 + 1 - 2e}{e}\right) \\
 &= \frac{2}{e}(e - 1)^2 \quad \text{Ans.}
 \end{aligned}$$

Q.15 $\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1} d\theta$



Solution:

$$\begin{aligned}
 &\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1} d\theta \\
 &= \int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{2\cos^2\theta} d\theta \quad \left(\because \cos 2\theta = 2\cos^2\theta - 1 \right. \\
 &\quad \left. 1 + \cos 2\theta = 2\cos^2\theta \right) \\
 &= \frac{1}{2} \int_0^{\pi/4} \left(\frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta} \right) d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{\cos\theta} + \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} \right) d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} \sec\theta d\theta + \int_0^{\pi/4} \sec\theta \tan\theta d\theta \\
 &= \frac{1}{2} [\ln |\sec\theta + \tan\theta|]_0^{\pi/4} + [\sec\theta]_0^{\pi/4} \\
 &= \frac{1}{2} [\ln |\sec\frac{\pi}{4} + \tan\frac{\pi}{4}| - \ln |\sec 0 + \tan 0| + \sec\frac{\pi}{4} - \sec 0] \\
 &= \frac{1}{2} [\ln (\sqrt{2} + 1) - \ln (1 + 0) + \sqrt{2} - 1] \\
 &= \frac{1}{2} [\ln (\sqrt{2} + 1) + \sqrt{2} - 1] \quad \text{Ans.}
 \end{aligned}$$

$$\text{Q.16} \quad \int_0^{\frac{\pi}{6}} \cos^3 \theta \, d\theta$$

Solution:

$$\begin{aligned}
 & \int_0^{\frac{\pi}{6}} \cos^3 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{6}} \cos \theta \cdot \cos^2 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{6}} \cos \theta (1 - \sin^2 \theta) \, d\theta \\
 &= \int_0^{\frac{\pi}{6}} \cos \theta d\theta - \int_0^{\frac{\pi}{6}} \sin^2 \theta \cos \theta d\theta \\
 &= [\sin \theta]_0^{\pi/6} - \left[\frac{\sin^3 \theta}{3} \right]_0^{\pi/6} \\
 &= \sin \frac{\pi}{6} - \sin 0 - \frac{1}{3} \left(\sin^3 \frac{\pi}{6} - \sin^3 0 \right) \\
 &= \frac{1}{2} - 0 - \frac{1}{3} \left[\left(\frac{1}{2} \right)^3 - 0 \right] \\
 &= \frac{1}{2} - \frac{1}{3} \left(\frac{1}{8} \right) = \frac{1}{2} - \frac{1}{24} \\
 &= \frac{12 - 1}{24} = \frac{11}{24} \quad \text{Ans.}
 \end{aligned}$$

$$\text{Q.17} \quad \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 \theta \cot^2 \theta \, d\theta$$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 \theta \cot^2 \theta \, d\theta$$

$$\begin{aligned}
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos\theta (\cosec^2\theta - 1) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos^2\theta \cosec^2\theta - \cos^2\theta) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos^2\theta}{\sin^2\theta} - \frac{1 + \cos2\theta}{2} \right) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\cot^2\theta - \frac{1 + \cos2\theta}{2} \right) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\cosec^2\theta - 1 - \frac{1 + \cos2\theta}{2} \right) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{2\cosec^2\theta - 2 - 1 - \cos2\theta}{2} \right) d\theta \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\cosec^2\theta - 3 - \cos2\theta) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cosec^2\theta d\theta - \frac{3}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta - \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos2\theta d\theta \\
&= [-\cot\theta]_{\pi/6}^{\pi/4} - \frac{3}{2} [\theta]_{\pi/6}^{\pi/4} - \frac{1}{2} \left[\frac{\sin2\theta}{2} \right]_{\pi/6}^{\pi/4} \\
&= -[\cot\frac{\pi}{4} - \cot\frac{\pi}{6}] - \frac{3}{2} \left(\frac{\pi}{4} - \frac{\pi}{6} \right) - \frac{1}{4} [\sin2\left(\frac{\pi}{4}\right) - \sin2\left(\frac{\pi}{6}\right)] \\
&= -1 + \sqrt{3} - \frac{3}{2} \left(\frac{3\pi - 2\pi}{12} \right) - \frac{1}{4} \left(1 - \frac{\sqrt{3}}{2} \right) \\
&= -1 + \sqrt{3} - \frac{\pi}{8} - \frac{1}{4} + \frac{\sqrt{3}}{8}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{-8 + 8\sqrt{3} - \pi - 2 + \sqrt{3}}{8} \\
 &= \frac{-10 - \pi + 9\sqrt{3}}{8} \quad \text{Ans.}
 \end{aligned}$$

Q.18 $\int_0^{\frac{\pi}{4}} \cos^4 t dt$

Solution:

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \cos^4 t dt \\
 &= \int_0^{\frac{\pi}{4}} (\cos^2 t)^2 dt \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1 + \cos 2t}{2}\right)^2 dt \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{4}} (1 + \cos^2 2t + 2\cos 2t) dt \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left(1 + \frac{1 + \cos 4t}{2} + 2\cos 2t\right) dt \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left(\frac{2 + 1 + \cos 4t + 4\cos 2t}{2}\right) dt \\
 &= \frac{1}{8} \int_0^{\frac{\pi}{4}} (3 + \cos 4t + 4\cos 2t) dt \\
 &= \frac{1}{8} \left[3 \int_0^{\frac{\pi}{4}} dt + \int_0^{\frac{\pi}{4}} \cos 4t dt + 4 \int_0^{\frac{\pi}{4}} \cos 2t dt \right] \\
 &= \frac{1}{8} \left[3 [t]_0^{\frac{\pi}{4}} + \left[\frac{\sin 4t}{4} \right]_0^{\frac{\pi}{4}} + 4 \left[\frac{\sin 2t}{2} \right]_0^{\frac{\pi}{4}} \right]
 \end{aligned}$$

$$\begin{aligned}
 \because \cos 2t &= 2 \cos^2 t - 1 \\
 2\cos^2 t &= 1 + \cos 2t \\
 \cos^2 t &= \frac{1 + \cos 2t}{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \cos^2 t &= \frac{1 + \cos 2t}{2} \\
 \cos^2 2t &= \frac{1 + \cos 4t}{2}
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{8} \left[3 \left(\frac{\pi}{4} - 0 \right) + \frac{1}{4} \left(\sin 4 \cdot \frac{\pi}{4} - \sin 0 \right) + 2 \left(\sin 2 \cdot \frac{\pi}{4} - \sin 0 \right) \right] \\
 &= \frac{1}{8} \left[\frac{3\pi}{4} + \frac{1}{4} (0 - 0) + 2 (1 - 0) \right] \\
 &= \frac{1}{8} \left(\frac{3\pi}{4} + 2 \right) \\
 &= \frac{1}{8} \left(\frac{3\pi + 8}{4} \right) \\
 &= \frac{3\pi + 8}{32} \quad \text{Ans.}
 \end{aligned}$$

Q.19 $\int_0^{\frac{\pi}{3}} \cos^2 \theta \sin \theta \, d\theta$

Solution:

$$\int_0^{\frac{\pi}{3}} \cos^2 \theta \sin \theta \, d\theta$$



$$\begin{aligned}
 &= - \int_0^{\frac{\pi}{3}} \cos^2 \theta (-\sin \theta) \, d\theta \\
 &= - \left[\frac{\cos^3 \theta}{3} \right]_0^{\pi/3} \\
 &= - \frac{1}{3} \left[\cos^3 \frac{\pi}{3} - \cos^3 0 \right] \\
 &= - \frac{1}{3} \left[\left(\frac{1}{2} \right)^3 - 1 \right] \\
 &= - \frac{1}{3} \left(\frac{1}{8} - 1 \right) \\
 &= - \frac{1}{3} \left(\frac{1 - 8}{8} \right) \\
 &= - \frac{1}{3} \left(\frac{-7}{8} \right) \\
 &= \frac{7}{24} \quad \text{Ans.}
 \end{aligned}$$

$$\text{Q.20} \quad \int_0^{\frac{\pi}{4}} (1 + \cos^2\theta) \tan^2\theta d\theta$$

Solution:

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} (1 + \cos^2\theta) \tan^2\theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta \cos^2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1 + \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1 + \sin^2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1 + \frac{1 - \cos 2\theta}{2}) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{2\sec^2\theta - 2 + 1 - \cos 2\theta}{2} \right) d\theta \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (2 \sec^2\theta - 1 - \cos 2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \sec^2\theta d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta \\
 &= [\tan\theta]_0^{\frac{\pi}{4}} - \frac{1}{2} [\theta]_0^{\frac{\pi}{4}} - \frac{1}{2} \left[\frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{4}} \\
 &= \left(\tan \frac{\pi}{4} - \tan 0 \right) - \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) - \frac{1}{4} \left[\sin 2 \left(\frac{\pi}{4} - \sin 2(0) \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= 1 - 0 - \frac{\pi}{8} - \frac{1}{4}(1 - 0) = 1 - \frac{\pi}{8} - \frac{1}{4} \\
 &= \frac{8 - \pi - 2}{8} = \frac{6 - \pi}{8} \quad \text{Ans.}
 \end{aligned}$$

Q.21 $\int_0^{\frac{\pi}{4}} \frac{\sec \theta}{\sin \theta + \cos \theta} d\theta$ (Guj. Board 2008)

Solution:

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \frac{\sec \theta}{\sin \theta + \cos \theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{1/\cos^2 \theta}{\tan \theta + 1} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\tan \theta + 1} d\theta \\
 &= [\ln |\tan \theta + 1|]_0^{\pi/4} = \ln |\tan \frac{\pi}{4} + 1| - \ln |\tan 0 + 1| \\
 &= \ln |1 + 1| - \ln |0 + 1| = \ln 2 - 0 = \ln 2 \quad \text{Ans.}
 \end{aligned}$$

Q.22 $\int_{-1}^5 |x - 3| dx$ (Lhr. Board 2009 (S))

Solution:

We know that

$$\begin{aligned}
 |x - 3| &= \begin{cases} + (x - 3) & \text{If } x > 3 \\ - (x - 3) & \text{If } x < 3 \\ 0 & \text{If } x = 3 \end{cases} \\
 &= \int_{-1}^5 |x - 3| dx + \int_3^5 (x - 3) dx \\
 &= - \int_{-1}^3 |(x - 3)| dx + \int_3^5 (x - 3) dx \\
 &= - \left[\frac{(x-3)^2}{2} \right]_{-1}^3 + \left[\frac{(x-3)^2}{2} \right]_3^5
 \end{aligned}$$

$$\begin{aligned}
 &= -\left[0 - \frac{16}{2} \right] + \left[\frac{4}{2} - 0 \right] = \frac{16}{2} + \frac{4}{2} = 8 + 2 = 10 \\
 \text{Q.23} \quad &\int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{1}{3}}} dx
 \end{aligned}$$

Solution:

$$\begin{aligned}
 &\int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{1}{3}}} dx \\
 &= 3 \int_{\frac{1}{8}}^1 (x^{\frac{1}{3}} + 2)^2 \cdot \frac{1}{3} x^{-\frac{2}{3}} dx \\
 &= 3 \left[\frac{(x+2)^3}{3} \right]_{1/8}^1 \\
 &= (1+2)^3 - \left((1/8)^{\frac{1}{3}} + 2 \right)^3 \\
 &= 27 - \left(\frac{1}{(2^3)^{\frac{1}{3}}} + 2 \right)^3 \\
 &= 27 - \left(\frac{1}{2} + 2 \right)^3 = 27 - \left(\frac{1+4}{2} \right)^3 \\
 &= 27 - \frac{125}{8} = \frac{216 - 125}{8} = \frac{91}{8} \quad \text{Ans.}
 \end{aligned}$$

$$\text{Q.24} \quad \int_1^3 \frac{x^2 - 2}{x + 1} dx$$

Solution:

$$\begin{aligned}
 &\int_1^3 \frac{x^2 - 2}{x + 1} dx \\
 &= \int_1^3 \left(x - 1 - \frac{1}{x+1} \right) dx \\
 &= \int_1^3 x dx - \int_1^3 dx - \int_1^3 \frac{dx}{x+1} \\
 &\qquad\qquad\qquad x+1 \sqrt{x^2-2} \\
 &\qquad\qquad\qquad \frac{-x^2 \pm x}{-x-2}
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{x^2}{2} \right]_1^3 - [x]_1^3 - [\ln|x+1|]_1^3 \\
 &= \frac{1}{2}(9-1) - (3-1) - [\ln|3+1| - \ln|1+1|] \\
 &= \frac{8}{2} - 2 - (\ln 4 - \ln 2) \\
 &= 4 - 2 - \ln\left(\frac{4}{2}\right) \\
 &= 2 - \ln 2 \quad \text{Ans.}
 \end{aligned}$$

Q.25 $\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} dx$

Solution:

$$\begin{aligned}
 &\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} dx \\
 &= \int_2^3 \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1} dx = [\ln|x^3 - x^2 + x - 1|]_2^3 \\
 &= \ln|27 - 9 + 3 - 1| - \ln|8 - 4 + 2 - 1| \\
 &= \ln 20 - \ln 5 = \ln \frac{20}{5} = \ln 4 \quad \text{Ans.}
 \end{aligned}$$

Q.26 $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x} dx$

Solution:

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x} dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \sec x \tan x dx - \int_0^{\frac{\pi}{4}} \sec^2 x dx \\
 &= [\sec x]_0^{\frac{\pi}{4}} - [\tan x]_0^{\frac{\pi}{4}}
 \end{aligned}$$

$$\begin{aligned}
 &= (\sec \frac{\pi}{4} - \sec 0) - (\tan \frac{\pi}{4} - \tan 0) \\
 &= \sqrt{2} - 1 - (1 - 0) = \sqrt{2} - 1 - 1 = \sqrt{2} - 2 \quad \text{Ans.}
 \end{aligned}$$

Q.27 $\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin x} dx$

Solution:

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin x} dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1 - \sin x}{1 - \sin^2 x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{1 - \sin x}{\cos^2 x} dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \sec^2 x dx - \int_0^{\frac{\pi}{4}} \sec x \tan x dx \\
 &= [\tan x]_0^{\frac{\pi}{4}} - [\sec x]_0^{\frac{\pi}{4}} \\
 &= \left(\tan \frac{\pi}{4} - \tan 0 \right) - \left(\sec \frac{\pi}{4} - \sec 0 \right) \\
 &= (1 - 0) - (\sqrt{2} - 1) = 1 - \sqrt{2} + 1 = 2 - \sqrt{2} \quad \text{Ans.}
 \end{aligned}$$

Q.28 $\int_0^1 \frac{3x}{\sqrt{4 - 3x}} dx$ (Lhr. Board 2008)

Solution:

$$\int_0^1 \frac{3x}{\sqrt{4 - 3x}} dx$$

$$\begin{aligned}
&= - \int_0^1 \frac{4 - 3x - 4}{\sqrt{4 - 3x}} dx \\
&= - \int_0^1 \left(\frac{4 - 3x}{\sqrt{4 - 3x}} - \frac{4}{\sqrt{4 - 3x}} \right) dx \\
&= - \int_0^1 \sqrt{4 - 3x} dx + 4 \int_0^1 \frac{dx}{\sqrt{4 - 3x}} \\
&= \frac{1}{3} \int_0^1 (4 - 3x)^{1/2} - 3dx - \frac{4}{3} \int_0^1 (4 - 3x)^{-1/2} - 3dx \\
&= \frac{1}{3} \left[\frac{(4 - 3x)^{3/2}}{\frac{3}{2}} \right] - \frac{4}{3} \left[\frac{(4 - 3x)^{1/2}}{\frac{1}{2}} \right]_0^1 \\
&= \frac{2}{9} \left[(4 - 3)^{\frac{3}{2}} - (4 - 0)^{\frac{3}{2}} \right] - \frac{8}{3} \left[(4 - 3)^{\frac{1}{2}} - (4 - 0)^{\frac{1}{2}} \right] \\
&= \frac{2}{9} [1 - (2^2)^{\frac{3}{2}}] - \frac{8}{3} [1 - (2^2)^{\frac{1}{2}}] \\
&= \frac{2}{9} (1 - 8) - \frac{8}{3} (1 - 2) \\
&= \frac{2}{9} (-7) - \frac{8}{3} (-1) = \frac{-14}{9} + \frac{8}{3} = \frac{-14 + 24}{9} = \frac{10}{9} \quad \text{Ans.}
\end{aligned}$$

Q.29 $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x (2 + \sin x)} dx$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x (2 + \sin x)} dx$$

Put $\sin x = t$

$$\cos x dx = dt \Rightarrow dx = \frac{dt}{\cos x}$$

$$\text{When } x = \frac{\pi}{6}, \quad t = \frac{1}{2}$$

When $x = \frac{\pi}{2}$, $t = 1$

$$= \int_{\frac{1}{2}}^1 \frac{\cos x}{t(2+t)} \times \frac{dt}{\cos x} = \int_{\frac{1}{2}}^1 \frac{dt}{t(2+t)}$$

Taking

$$\frac{t}{t(2+t)} = \frac{A}{t} + \frac{B}{2+t} \quad \text{--- (1)}$$

Multiplying $t(2+t)$ on both sides in eq. (1)

$$1 = A(t+2) + Bt \quad \text{--- (2)}$$

To find A

Put $t = 0$ in equation (2)

$$1 = A(0+2)$$

$$2A = 1$$

$$A = \frac{1}{2}$$

To find B

Put $2+t = 0$

$t = -2$ in equation (2)

$$1 = B(-2)$$

$$B = \frac{-1}{2}$$



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∴ From equation (1)

$$\frac{1}{t(2+t)} = \frac{\frac{1}{2}}{t} + \frac{\frac{-1}{2}}{2+t}$$

Integrate from $\frac{1}{2}$ to 1

$$\begin{aligned} \int_{\frac{1}{2}}^1 \frac{dt}{t(2+t)} &= \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{dt}{t} - \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{dt}{2+t} \\ &= \frac{1}{2} [\ln|t|]_{\frac{1}{2}}^1 - \frac{1}{2} [\ln|2+t|]_{\frac{1}{2}}^1 \\ &= \frac{1}{2} \left(\ln 1 - \ln \frac{1}{2} \right) - \frac{1}{2} \left(\ln 3 - \ln \left(2 + \frac{1}{2} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \left[-\ln \frac{1}{2} - \ln 3 + \ln \left(\frac{4+1}{2} \right) \right] \\
 &= \frac{1}{2} \left[-\ln \frac{1}{2} - \ln 3 + \ln \frac{5}{2} \right] \\
 &= \frac{1}{2} \ln \left(\frac{5/2}{\frac{1}{2} \times 3} \right) \\
 &= \frac{1}{2} \ln \left(\frac{5}{3} \right) \quad \text{Ans}
 \end{aligned}$$

Q.30 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)} dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)} dx$$

Put

$$\begin{aligned}
 \cos x &= t \\
 -\sin x \, dx &= dt \\
 dx &= \frac{dt}{-\sin x}
 \end{aligned}$$

When $x = 0, t = 1$



When $t = \frac{\pi}{2}, t = 0$

$$= \int_1^0 \frac{\sin x}{(1+t)(2+t)} \times \frac{dt}{-\sin x} = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

Taking

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t} \quad (1)$$

Multiplying $(1+t)(2+t)$ on both sides in eq. (1)

$$1 = A(2+t) + B(1+t) \quad (2)$$

To find A

Put

$$1+t = 0$$

$$t = -1 \text{ in equation (2)}$$

$$1 = A(2-1)$$

$$\boxed{A = 1}$$

To find B

Put $2 + t = 0$

$$t = -2 \text{ in equation (2)}$$

$$1 = B(1 - 2)$$

$$-B = 1$$

$$\boxed{B = -1}$$

\therefore From equation (1)

$$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} + \frac{-1}{2+t}$$

Integrate from 0 to 1

$$\begin{aligned} \int_0^1 \frac{dt}{(1+t)(2+t)} &= \int_0^1 \frac{dt}{1+t} - \int_0^1 \frac{dt}{2+t} \\ &= [\ln|1+t|]_0^1 - [\ln|2+t|]_0^1 \\ &= (\ln 2 - \ln 1) - (\ln 3 - \ln 2) \\ &= \ln 2 - \ln 3 + \ln 2 \\ &= \ln \frac{2 \times 2}{3} = \ln \frac{4}{3} \quad \text{Ans} \end{aligned}$$

EXERCISE 3.7

Q.1 Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$
(Lhr. Board 2005, 2008)

Solution:

$$y = x^2 + 1 \quad \text{from } x = 1 \text{ to } x = 2$$

$$\text{Required area} = \int_b^a y dx$$

$$= \int_{\frac{1}{2}}^{\frac{2}{2}} (x^2 + 1) dx$$

$$= \int_{\frac{1}{2}}^{\frac{2}{2}} x^2 dx + \int_{\frac{1}{2}}^{\frac{2}{2}} dx$$

$$= \left[\frac{x^3}{3} \right]_1^2 + [x]_1^2$$