EXERCISE 4.5

Q:1 Find the lines represented by each of the following and also find measure of the angle between them (Problems 1 – 6).

1. $\mathbf{10x^2 - 23xy - 5y^2 = 0}$

2. $\mathbf{3x^2 + 7xy + 2y^2 = 0}$ **(Lhr. Board 2009 (S))**

3. $\mathbf{9x^2 + 24xy + 16y^2 = 0}$

4. $\mathbf{2x^2 + 3xy - 5y^2 = 0}$

5. $\mathbf{6x^2 - 19xy + 15y^2 = 0}$

6. $\mathbf{x^2 + 2xy \sec\alpha + y^2 = 0}$

Solution:

$10x^2 - 23xy - 5y^2 = 0$ **(Guj. Board 2007)**

$10x^2 - 25xy + 2xy - 5y^2 = 0$

$5x(2x - 5y) + y(2x - 5y) = 0$

Either

$2x - 5y = 0$ or $5x + y = 0$

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$

$a = 10$, $2h = -23$, $b = -5$

$h = \dfrac{-23}{2}$

Let θ be an angle between the lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{-23}{2}\right)^2 - (10)(-5)}}{10 - 5}$$

$$\tan\theta = \frac{2\sqrt{\frac{529}{4} + 50}}{5}$$

$$\tan\theta = \frac{2\sqrt{\frac{529 + 200}{4}}}{5}$$

$$\tan\theta = \frac{2\sqrt{\frac{729}{4}}}{5}$$

$$\tan\theta = \frac{2\left(\frac{27}{2}\right)}{5}$$

$$\tan\theta = \frac{27}{5}$$

$$\theta = \tan^{-1}\left(\frac{27}{5}\right)$$

$$\boxed{\theta = 79.51^{\circ}}$$ Ans.

2. $3x^2 + 7xy + 2y^2 = 0$ (Guj. Board 2006) (Lhr. Board 2006)

$$3x^2 + 6xy + xy + 2y^2 = 0$$

$$3x(x + 2y) + y(x + 2y) = 0$$

$$(x + 2y)(3x + y) = 0$$

Either

$x + 2y = 0$ or $3x + y = 0$

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$

$a = 3$, $2h = 7$, $b = 2$

$$h = \frac{7}{2}$$

Let θ be an angle between the lines.

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - (3)(2)}}{3 + 2}$$

$$\tan\theta = \frac{2\sqrt{\frac{49}{4} - 6}}{5}$$

$$\tan\theta = \frac{2\sqrt{\frac{25}{4}}}{5}$$

$$\tan\theta = \frac{2\left(\frac{5}{2}\right)}{5}$$

$$\tan\theta = \frac{5}{5}$$

$$\tan\theta = 1$$

$$\theta = \tan^{-1}(1)$$

$$\boxed{\theta = 45^{o}}\quad \text{Ans}$$

3. $9x^2 + 24xy + 16y^2 = 0$

$9x^2 + 12xy + 12xy + 16y^2 = 0$

$3x(3x + 4y) + 4y(3x + 4y) = 0$

$(3x + 4y)(3x + 4y) = 0$

$3x + 4y = 0$

is the equation of two coincident lines passing through origin.

4. $2x^2 + 3xy - 5y^2 = 0$

$2x^2 + 5xy - 2xy - 5y^2 = 0$

$x(2x + 5y) - y(2x + 5y) = 0$

$(2x + 5y)(x - y) = 0$

Either

$2x + 5y = 0$ or $x - y = 0$

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$

$a = 2$, $2h = 3$, $b = -5$

$$h = \frac{3}{2}$$

Let θ be an angle between the lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(-5)}}{2 - 5}$$

$$\tan\theta = \frac{2\sqrt{\frac{9}{4} + 10}}{-3} = \frac{2\sqrt{\frac{9 + 40}{4}}}{-3}$$

$$\tan\theta = \frac{-2\sqrt{\frac{49}{4}}}{3}$$

$$\tan\theta = \frac{-2\left(\frac{7}{2}\right)}{3}$$

$$\tan\theta = \frac{-7}{3}$$

$$-\tan\theta = \frac{7}{3}$$

$$\tan(180^o - \theta) = \frac{7}{3}$$

$$180 - \theta = \tan^{-1}\left(\frac{7}{3}\right)$$

$$180 - \theta = 66.80^o$$

$$\theta = 180^o - 66.80^o$$

$$\boxed{\theta = 113.2^o} \quad \text{Ans.}$$

5. $$6x^2 - 19xy + 15y^2 = 0$$

$$6x^2 - 10xy - 9xy + 15y^2 = 0$$

$$2x(3x - 5y) - 3y(3x - 5y) = 0$$

$$(3x - 5y)(2x - 3y) = 0$$

Either

$$3x - 5y = 0 \quad \text{or} \quad 2x - 3y = 0$$

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$

$a = 6$, $2h = -19$, $b = 15$

$$h = \frac{-19}{2}$$

Let ‘θ’ be an angle between the lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{-19}{2}\right)^2 - (6)(15)}}{6 + 15}$$

$$\tan\theta = \frac{2\sqrt{\frac{361}{4} - 90}}{21}$$

$$\tan\theta = \frac{2\sqrt{\frac{361 - 360}{4}}}{21}$$

$$\tan\theta = \frac{2\sqrt{\frac{1}{4}}}{21}$$

$$\tan\theta = \frac{2\left(\frac{1}{2}\right)}{21}$$

$$\tan\theta = \frac{1}{21}$$

$$\theta = \tan^{-1}\left(\frac{1}{21}\right)$$

$$\boxed{\theta = 2.73^\circ} \qquad \text{Ans}$$

6. $x^2 + 2xy\sec\alpha + y^2 = 0$

Dividing both sides by x^2

$$\frac{y^2}{x^2} + \frac{2xy\sec\alpha}{x^2} + \frac{x^2}{x^2} = \frac{0}{x^2}$$

$$\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)\sec\alpha + 1 = 0$$

Equation is quadratic in $\frac{y}{x}$

So,

$a = 1$, $b = 2\sec\alpha$, $c = 1$

$$\frac{y}{x} = \frac{-2\sec\alpha \pm \sqrt{(2\sec\alpha)^2 - 4(1)(1)}}{2(1)}$$

$$\frac{y}{x} = \frac{-2\sec\alpha \pm \sqrt{\sec^2\alpha - 1}}{2}$$

$$\frac{y}{x} = -\sec\alpha \pm \tan\alpha$$

$$\frac{y}{x} = -\frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha}$$

$$\frac{y}{x} = \frac{-1 \pm \sin\alpha}{\cos\alpha}$$

$$\boxed{x(1 - \sin\alpha) + y\cos\alpha = 0}$$

$$\boxed{x(1 + \sin\alpha) + y\cos\alpha = 0}$$

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$

$a = 1,\quad 2h = 2\sec\alpha,\quad b = 1$

$h = \sec\alpha$

Let θ be an angle between the lines.

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\sec^2\alpha - (1)(1)}}{1 + 1}$$

$$\tan\theta = \frac{2\sqrt{\sec^2\alpha - 1}}{2}$$

$$\tan\theta = \sqrt{\tan^2\alpha}$$

$$\tan\theta = \tan\alpha$$

$$\boxed{\theta = \alpha} \quad \text{Ans.}$$

Q.7 Find a joint equation of the lines through the origin and perpendicular to the lines $x^2 - 2xy\tan\alpha - y^2 = 0$

Solution:

$$x^2 - 2xy\tan\alpha - y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$$a = 1, \quad 2h = -2\tan\alpha, \quad b = -1$$

$$h = -\tan\alpha$$

Let m_1, m_2 be the slopes of given lines.

$$m_1 + m_2 = \frac{-2h}{b}$$

$$= \frac{-2(-\tan\alpha)}{-1} = -2\tan\alpha$$

$$m_1 m_2 = \frac{a}{b}$$

$$= \frac{1}{-1} = -1$$

Slopes of lines perpendicular to given lines are $\frac{-1}{m_1}$ and $\frac{-1}{m_2}$

$$y = \frac{-1}{m_1}x \quad \text{and} \quad y = -\frac{1}{m_2}x$$

$$m_1 y = -x \quad \text{and} \quad m_2 y = -x$$

$$x + m_1 y = 0 \quad \text{and} \quad x + m_2 y = 0$$

So the joint eq. is

$$(x + m_1 y)(x + m_2 y) = 0$$

$$x^2 + m_2 xy + m_1 xy + m_1 m_2 y^2 = 0$$

$$x^2 + (m_1 + m_2)\,xy + m_1m_2y^2 = 0$$
$$x^2 + (-2\tan\alpha)\,xy + (-1)y^2 = 0$$
$$\boxed{x^2 - 2\tan\alpha\ xy - y^2 = 0}\quad \text{Ans}$$

Q.8 Find a joint equation of the lines through the origin and perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$

Solution:

$$ax^2 + 2hxy + by^2 = 0$$

Let m_1, m_2 be the slopes of given lines.

$$m_1 + m_2 = \frac{-2h}{b}$$

$$m_1\,m_2 = \frac{a}{b}$$

Slopes of lines perpendicular to given lines are $\frac{-1}{m_1}$ and $\frac{-1}{m_2}$ then their equations are

$$y = \frac{-1}{m_1}x \quad \text{and} \quad y = \frac{-1}{m_2}x$$
$$m_1 y = -x \quad \text{and} \quad m_2y = -x$$
$$x + m_1y = 0 \quad \text{and} \quad x + m_2y = 0$$

So the joint equation is

$$(x + m_1y)(x + m_2y) = 0$$
$$x^2 + m_2yx + m_1yx + m_1m_2y^2 = 0$$
$$x^2 + (m_1 + m_2)\,xy + m_1m_2y^2 = 0$$
$$x^2 - \frac{2h}{b}xy + \frac{a}{b}y^2 = 0$$
$$\frac{bx^2 - 2hxy + ay^2}{b} = 0$$
$$\boxed{bx^2 - 2hxy + ay^2 = 0}\quad \text{Ans.}$$

Q.9 Find the area of the region bounded by

$10x^2 - xy - 21y^2 = 0$ and $x + y + 1 = 0$

Solution:

$$10x^2 - xy - 21y^2 = 0 \quad \text{..... (1)}$$
$$x + y + 1 = 0 \quad \text{..... (2)}$$

From equation (1)

$$10x^2 - xy - 21y^2 = 0$$
$$10x^2 - 15xy + 14xy - 21y^2 = 0$$

$5x(2x - 3y) + 7y(2x - 3y) = 0$

$(2x - 3y)(5x + 7y) = 0$

$2x - 3y = 0$ ..... (3) or $5x + 7y = 0$ ...... (4)

Let A (x, y) be the point of intersection of equation (3) and equation (4) since equation (3) and equation (4) passing through origin so the point of intersection of (3) and (4) is A (0, 0).

Let B (x, y) be the point of intersection of equation (2) and (3).

Equation (2) × 3 + Equation (3), we get

$$\begin{array}{rl} 3x + 3y + 3 &= 0 \\ 2x - 3y &= 0 \\ \hline 5x + 3 &= 0 \end{array}$$

$5x = -3$

$x = \dfrac{-3}{5}$

Put $x = \dfrac{-3}{5}$ in eq. (3)

$2\left(\dfrac{-3}{5}\right) - 3y = 0$

$\dfrac{-6}{5} = 3y$

$y = \dfrac{-6}{15} = \dfrac{-2}{5}$

$\therefore$ $B(x, y) = B\left(\dfrac{-3}{5}, \dfrac{-2}{5}\right)$

Let C (x, y) be the point of intersection of equation (2) and equation (4)

Equation (2) × 5 – Equation (4), we get

$$\begin{array}{rl} 5x + 5y + 5 &= 0 \\ \underset{-}{}5x \underset{\pm}{} 7y &= 0 \\ \hline -2y + 5 &= 0 \end{array}$$

$-2y = -5$

$y = \dfrac{5}{2}$

Put $y = \dfrac{5}{2}$ in equation (4)

$5x + 7\left(\dfrac{5}{2}\right) = 0$

$$5x + \frac{35}{2} = 0$$

$$5x = \frac{-35}{2}$$

$$x = \frac{-35}{10} = \frac{-7}{2}$$

$$\therefore \quad C(x, y) = C\left(\frac{-7}{2}, \frac{5}{2}\right)$$

Area of region ABC is

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{-3}{5} & \frac{-2}{5} & 1 \\ \frac{-7}{2} & \frac{5}{2} & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \frac{-3}{5} & \frac{-2}{5} \\ \frac{-7}{2} & \frac{5}{2} \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{-15}{10} - \frac{14}{10}\right) = \frac{1}{2}\left(\frac{-15-14}{10}\right)$$

$$= \frac{1}{2}\left(\frac{-29}{10}\right) = \frac{29}{20}$$ Sq. unit (Neglecting – ve sign) Ans