Chapter 6

CONIC SECTION

Conic Sections or simply conics are the curves obtained by cutting a right circular cone by a plane.

If the cone is cut by a plane perpendicular to the axis of the cone, then the section is a circle.

Definition of a Circle

(Lahore Board 2009, 2011)

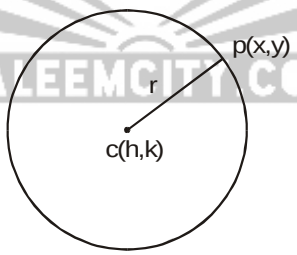
The set of all the points in the plane that are equally distant from a fixed point is called a circle.

The fix point is called center of circle. The distance from a center to any point of circle is called radius of circle.

Equation of Circle

If c (h, k) be center of a circle and p(x, y) be any point on the circle.

'r' is distance from center to any point of circle i.e. r is radius of circle.



By distance formula

$|cp| = \sqrt{(x-h)^2 + (y-k)^2}$

$r = \sqrt{(x-h)^2 + (y-k)^2}$ $\quad (\because |cp| = r)$

$r^2 = (x-h)^2 + (y-k)^2$ ..... (I)

is an equation of circle of standard form.

If center of circle is origin i.e; c(0, 0) then equation of circle becomes

$r^2 = (x-0)^2 + (y-0)^2$

$r^2 = x^2 + y^2$

Point Circle:

If $r = 0$ then circle is called point circle.

Parametric Equations of Circle:

$x = r\cos\theta, \qquad y = r\sin\theta$

General Form of an equation of Circle

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle g, f, c being constants.

$x^2 + y^2 + 2gx + 2fy + c = 0$

$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$ ($\because$ Adding g^2 and f^2 on both sides)

$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

$(x + g^2) + (y + f)^2 = g^2 + f^2 - c$

$(x - (-g))^2 + (x - (-f))^2 = g^2 + f^2 - c$

Which is standard form of an equation of circle, where center is (–g, –f) and radius is $\sqrt{g^2 + f^2 - c}$.

EXERCISE 6.1

Q.1: In each of following find an equation of circle with

(a) Center at (5,–2) and radius 4.
(Lahore Board 2009)

Solution:

The equation of circle by standard form is

$(x - h)^2 + (y - k)^2 = r^2$

$(x - 5)^2 + (y - (-2))^2 = (4)^2$

$(x - 5)^2 + (y + 2)^2 = 16$

$x^2 + 25 - 10x + y^2 + 4 + 4y - 16 = 0$

$x^2 + y^2 - 10x + 4y + 13 = 0$

(b) Center at $(\sqrt{2}, -3\sqrt{3})$ radius $2\sqrt{2}$

Solution:

The equation of circle by standard form is

$(x - h)^2 + (y - k)^2 = r^2$

$(x - \sqrt{2})^2 + (y - (-3\sqrt{3}))^2 = (2\sqrt{2})^2$

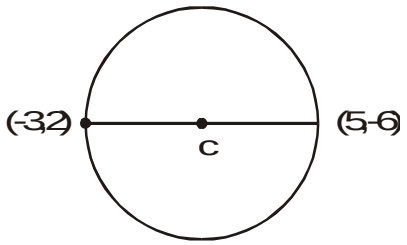
$x^2 + 2 - 2\sqrt{2}\,x + y^2 + 27 + 6\sqrt{3}\,y = 8$

$x^2 + y^2 - 2\sqrt{2}\,x + 6\sqrt{3}\,y + 29 - 8 = 0$

$x^2 + y^2 - 2\sqrt{2}\,x + 6\sqrt{3}\,y + 21 = 0$

(c) ends of a diameter at (–3, 2) & (5, –6)

Solution:



Center = Mid point of ends of diameter

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{Center} = \left(\frac{-3+5}{2}, \frac{2-6}{2}\right) = \left(\frac{2}{2}, \frac{-4}{2}\right) = (1, -2)$$

Radius = Distance from center to any one point of circle

$$= \sqrt{(1+3)^2 + (-2-2)^2}$$

$$= \sqrt{16 + 16}$$

$$= \sqrt{32}$$

The equation of circle by standard form

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-1)^2 + (y-(-2))^2 = (\sqrt{32})^2$$

$$(x-1)^2 + (y+2)^2 = 32$$

$$x^2 + 1 - 2x + y^2 + 4 + 4y - 32 = 0$$

$$x^2 + y^2 - 2x + 4y - 27 = 0$$

Q.2: Find center & radius of the circle with the given equation

(a) $x^2 + y^2 + 12x - 10y = 0$

(Lahore Board 2010)

Solution:

General from of an equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Compare it with $x^2 + y^2 + 12x - 10y = 0$

$2g = 12$, $2f = -10$ $c = 0$

$g = 6$, $f = -5$ $c = 0$

We know that

Center of general form is $(-g, -f) = (-6, 5)$

Radius of circle is $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(6)^2 + (-5)^2 - 0}$$

$$= \sqrt{36 + 25}$$
$$= \sqrt{61}$$

(b) $\mathbf{5x^2 + 5y^2 + 14x + 12y - 10 = 0}$

Solution:

$5x^2 + 5y^2 + 14x + 12y - 10 = 0$

Dividing throughout by 5

$$\frac{5x^2}{5} + \frac{5y^2}{5} + \frac{14x}{5} + \frac{12y}{5} - \frac{10}{5} = 0$$

$$x^2 + y^2 + \frac{14x}{5} + \frac{12y}{5} - 2 = 0$$

Comparing with general form of an equation of circle

$$2g = \frac{14}{5}, \quad 2f = \frac{12}{5}, \quad c = -2$$

$$g = \frac{7}{5}, \quad f = \frac{6}{5}, \quad c = -2$$

Center of circle $= (-g, -f) = \left(-\frac{7}{5}, -\frac{6}{5}\right)$

Radius of circle $= \sqrt{(g^2 + f^2 - c)} = \sqrt{\left(\frac{7}{2}\right)^2 + \left(\frac{6}{5}\right)^2 - (-2)}$

$$= \sqrt{\frac{49}{25} + \frac{36}{25} + 2}$$

$$= \sqrt{\frac{49 + 36 + 50}{25}} = \sqrt{\frac{135}{25}}$$

$$= \frac{\sqrt{135}}{5} \text{ Ans.}$$

(c) $\mathbf{x^2 + y^2 - 6x + 4y + 13 = 0}$ **(Lahore Board 2009)**

Solution:

$x^2 + y^2 - 6x + 4y + 13 = 0$

Comparing it with general form of an equation of circle

$$2g = -6, \quad 2f = 4, \quad c = 13$$
$$g = -3, \quad f = 2, \quad c = 13$$

Center of circle $= (-g, -f) = (3, -2)$

Radius of circle $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(-3)^2 + (2)^2 - 13}$$
$$= \sqrt{9 + 4 - 13}$$

$= \sqrt{13 - 13}$

$= 0$

(d) $\mathbf{4x^2 + 4y^2 - 8x + 12y - 25 = 0}$

Solution:

$4x^2 + 4y^2 - 8x + 12y - 25 = 0$

Dividing throughout by 4

$x^2 + y^2 - 2x + 3y - \frac{25}{4} = 0$

Comparing it with general form of an equation of circle

$2g = -2 \qquad 2f = 3 \qquad c = -\frac{25}{4}$

$g = -1 \quad , \quad f = \frac{3}{2}$

Center of Circle $= (-g, -f) = (1, -\frac{3}{2})$

Radius of Circle $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{(-1)^2 + \left(\frac{3}{2}\right)^2 - \left(\frac{-25}{4}\right)}$

$= \sqrt{1 + \frac{9}{4} + \frac{25}{4}} = \sqrt{\frac{4 + 9 + 25}{4}}$

$= \sqrt{\frac{38}{4}} = \frac{\sqrt{38}}{2}$

Q.3: Write an equation of the circle that passes through the given points.

(a) A(4, 5) , B(–4, – 3) C(8, – 3)

(Lahore Board 2009, 2011)

Solution:

When the circle passes through different points we use general equation of circle.

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (I)$

For A (4, 5) we have

$(4)^2 + (5)^2 + 2g(4) + 2f(5) + c = 0$

$16 + 25 + 8g + 10f + c = 0$

$41 + 8g + 10f + c = 0 \qquad (i)$

For B (–4, –3) we have

$(-4)^2 + (-3)^2 + 2g(-4) + 2f(-3) + c = 0$

$16 + 9 - 8g - 6f + c = 0$

$25 - 8g - 6f + c = 0 \qquad (ii)$

For C (8, –3) we have

$$(8)^2 + (-3)^2 + 2g(8) + 2f(-3) + c = 0$$
$$64 + 9 + 16g - 6f + c = 0$$
$$73 + 16g - 6f + c = 0 \qquad \text{(iii)}$$

Subtracting (ii) form (i)

$$41 + 8g + 10f + c = 0$$
$$-\ 25 \mp 8g \mp 6f \pm c = 0$$
$$16 + 16g + 16f = 0$$
$$1 + g + f = 0 \qquad \text{(iv)}$$
$$1 - 2 + f = 0$$
$$1 - 2 + f = 0$$
$$-1 + f = 0 \quad \Rightarrow \quad \boxed{f = 1}$$

Subtracting (iii) form (ii)

$$25 - 8g - 6f + c = 0$$
$$-\ 73 \pm 16g \mp 6f \pm c = 0$$
$$-48 - 24g = 0$$
$$2 + g = 0 \qquad \boxed{g = -2}$$

Put in (iv)

Now putting values in (i)

$$41 + 8(-2) + 10(1) + c = 0$$
$$41 - 16 + 10 + c = 0$$
$$35 + c = 0$$
$$\boxed{c = -35}$$ Substitute all value in (I)

$$x^2 + y^2 + 2(-2)x + 2(1)y - 35 = 0$$
$$x^2 + y^2 - 4x + 2y - 35 = 0$$

(b) A(–7 , 7) B (5, –1) C(10, 0)

Solution:

General equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(I)}$$

For A(–7, 7) we have $(-7)^2 + (7)^2 + 2g(-7) + 2f(7) + c = 0$

$$49 + 49 - 14g + 14f + c = 0$$
$$98 - 14g + 14f + c = 0 \qquad \text{..... (i)}$$

For B(5, –1) $(5)^2 + (-1)^2 + 2g(5) + 2f(-1) + c = 0$

$$25 + 1 + 10g - 2f + c = 0$$
$$26 + 10g - 2f + c = 0 \qquad \text{..... (ii)}$$

For C(10, 0) $(10)^2 + (0)^2 + 2g(10) + 2f(0) + c = 0$

$$100 + 20g + c = 0 \qquad \text{..... (iii)}$$

Subtracting (ii) form (i)

$$98 - 14g + 14f + c = 0$$
$$-\ 26 \mp 10g \mp 2f \pm c = 0$$
$$72 - 24g + 16f = 0 \quad \text{(iv)}$$
$$8(9 - 3g + 2f) = 0$$
$$9 - 3g + 2f = 0 \quad \text{(vi)}$$

Subtracting (iii) form (ii)

$$26 + 10g - 2f + c = 0$$
$$-\ 100 \pm 20g \pm c = 0$$
$$-74 - 10g - 2f = 0 \quad \text{(v)}$$

Adding (v) & (vi)

$$-74 - 10g - 2f = 0$$
$$9 - 3g + 2f = 0$$
$$-65 - 13g = 0$$
$$-65 = 13g$$

$\boxed{g = -5}$ Put in (vi)

$$9 - 3(-5) + 2f = 0$$
$$9 + 15 + 2f = 0$$
$$2f = -24 \qquad \boxed{f = -12}$$

Put in (iii)

$$100 + 20(-5) + c = 0$$
$$100 - 100 + c = 0$$
$$\boxed{c = 0}$$

Substitute all values in (I)

$$x^2 + y^2 + 2(-5)x + 2(-12)y + 0 = 0$$
$$x^2 + y^2 - 10x - 24y = 0 \quad \text{Ans.}$$

(c) A(a, 0) , B(0, b) , C(0, 0)

Solution:

General equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(I)}$$

For A (a, 0) we have

$$a^2 + (0)^2 + 2g(a) + 2f(0) + c = 0$$
$$a^2 + 2ag + c = 0 \quad \text{(i)}$$

For B(0, b)

$$b^2 + 2fb + c = 0 \quad \text{(ii)}$$

For C(0, 0)

$$(0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0$$

$\Rightarrow \quad \boxed{c = 0}$

Putting value of c in (i) and (ii) we have

$a^2 + 2ag = 0$	$b^2 + 2fb = 0$
$a(a + 2g) = 0$	$b(b + 2f) = 0$
$a + 2g = 0$	$b + 2f = 0$
$2g = -a$	
$\Rightarrow \boxed{g = \frac{-a}{2}}$	$\boxed{f = \frac{-b}{2}}$

Substituting all values in (I) we have

$$x^2 + y^2 + 2\left(\frac{-a}{2}\right)x + 2\left(\frac{-b}{2}\right)y + c = 0$$

$$x^2 + y^2 - ax - by + c = 0 \qquad \text{Ans}$$

(d) A(5, 6) , B(– 3, 2) , C(3, –4)
(Gujranwala Board 2006)

Solution:

General equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(I)}$$

For A (5, 6) we have

$$(5)^2 + (6)^2 + 2g(5) + 2f(6) + c = 0$$
$$25 + 36 + 10g + 12f + c = 0$$
$$61 + 10g + 12f + c = 0 \qquad \text{(i)}$$

For B(– 3, 2)

$$(-3)^2 + (2)^2 + 2g(-3) + 2f(2) + c = 0$$
$$9 + 4 - 6g + 4f + c = 0$$
$$13 - 6g + 4f + c = 0 \qquad \text{(ii)}$$

For C(3, – 4)

$$(3)^2 + (-4)^2 + 2g(3) + 2f(-4) + c = 0$$
$$9 + 16 + 6g - 8f + c = 0$$
$$25 + 6g - 8f + c = 0 \qquad \text{(iii)}$$

Subtracting (ii) from (i) & (iii) from (ii)

$$\begin{array}{rl} & 61 + 10g + 12f + c = 0 \\ - & 13 \mp 6g \pm 4f \pm c = 0 \\ \hline & 48 + 16g + 8f = 0 \end{array} \qquad \begin{array}{rl} & 13 - 6g + 4f + c = 0 \\ - & 25 \pm 6g \mp 8f \pm c = 0 \\ \hline & -12 - 12g + 12f = 0 \end{array}$$

$6 + 2g + f = 0$ ..... (iv) (Dividing by 8) $\quad 1 + g - f = 0$ ..... (v)

Adding (iv) & (v)

$$\begin{array}{rl} & 6 + 2g + f = 0 \\ & 1 + g - f = 0 \\ \hline \end{array}$$

$$7 + 3g = 0$$

$$\boxed{g = \frac{-7}{3}}$$ Put in ..... (v) $1 - \frac{7}{3} - f = 0$

$$\frac{3-7}{3} = f$$

$$\boxed{\frac{-4}{3} = f}$$

Put in (ii)

$$13 - 6\left(\frac{-7}{3}\right) + 4\left(\frac{-4}{3}\right) + c = 0$$

$$13 + 14 - \frac{16}{3} + c = 0$$

$$65 + 3c = 0 \quad \boxed{c = \frac{-65}{3}}$$ substitute all values in (I)

$$x^2 + y^2 + 2\left(\frac{-7}{3}\right)x + 2\left(\frac{-4}{3}\right)y - \frac{65}{3} = 0$$

$$x^2 + y^2 - \frac{14}{3}x - \frac{8}{3}y - \frac{65}{3} = 0$$ Ans

Q.4: In each of the following, find an equation of the circle passing through

(a) A(3, –1) B(0, 1) and having center at 4x – 3y – 3 = 0

Solution:

We know that General equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(I)}$$

For A(3, –1) $(3)^2 + (-1)^2 + 2g(3) + 2f(-1) + c = 0$

$$9 + 1 + 6g - 2f + c = 0$$

$$10 + 6g - 2f + c = 0 \qquad \text{(i)}$$

For B(0, 1) $(0)^2 + (1)^2 + 2g(0) + 2f(1) + c = 0$

$$1 + 2f + c = 0 \qquad \text{(ii)}$$

Subtracting (ii) from (i)

$$\begin{array}{rl} & 10 + 6g - 2f + c = 0 \\ - & 1 \pm 0g \pm 2f \pm c = 0 \\ \hline & 9 + 6g - 4f = 0 \qquad \text{(iii)} \end{array}$$

Since center (–g, –f) lies at $4x - 3y - 3 = 0$

we have $4(-g) - 3(-f) - 3 = 0$

$$-4g + 3f - 3 = 0 \qquad \text{..... (iv)}$$

Multiplying equation (iii) by 3 and (iv) by 4 and adding

$$\begin{array}{rl} & 27 + 18g - 12f = 0 \\ - & 12 - 16g + 12f = 0 \\ \hline & 15 + 2g = 0 \end{array}$$

$$2g = -15$$

$$\boxed{g = \frac{-15}{2}}$$ Put in (iii)

$$9 + 6\left(\frac{-15}{2}\right) - 4f = 0$$

$$9 - 45 - 4f = 0$$

$$-36 = 4f$$

$$\boxed{f = -9}$$ Put in (ii)

$$1 + 2(-9) + c = 0$$

$$1 - 18 + c = 0 \qquad \boxed{c = 17}$$

Substitute values in (I)

$$x^2 + y^2 + 2\left(\frac{-15}{2}\right)x + 2(-9)y + 17 = 0$$

$$x^2 + y^2 + -15x - 18y + 17 = 0$$ Ans.

(b) A (– 3, 1) with radius 2 and center at 2x – 3y + 3 = 0

Solution:

With center (h, k) and radius r, we know equation of standard form of circle is

$$(x - h)^2 + (y - k)^2 = r^2$$

For A(–3, 1) $(-3 - h)^2 + (1 - k)^2 = r^2$

$$9 + h^2 + 6h + 1 + k^2 - 2k - 4 = 0$$

$$h^2 + k^2 + 6h - 2k + 6 = 0 \qquad \text{(i)}$$

Since center (h, k) lies at $2x - 3y + 3 = 0$

$$2h - 3k + 3 = 0$$

$$\Rightarrow \quad h = \frac{3k - 3}{2} \qquad \text{(ii)}$$

Put in (i)

$$\left(\frac{3k - 3}{2}\right)^2 + k^2 + 6\left(\frac{3k - 3}{2}\right) - 2k + 6 = 0$$

$$\frac{9k^2 + 9 - 18k}{4} + k^2 + 9k - 9 - 2k + 6 = 0$$

$$9k^2 + 9 - 18k + 4k^2 + 36k - 36 - 8k + 24 = 0$$

$$13k^2 + 10k - 3 = 0$$

$$13k^2 + 13k - 3k - 3 = 0$$

$$(13k - 3)\quad(k + 1) = 0$$

$$k = -1 \qquad\qquad k = \frac{3}{13}$$

Putting $k = -1$ in (ii)

$$h = \frac{3(-1) - 3}{2}$$

$$= \frac{-3 - 3}{2}$$

$$= \frac{-6}{2}$$

$$= -3$$

Putting $k = \frac{3}{13}$ in (ii)

$$h = \frac{3\left(\frac{3}{13}\right) - 3}{2}$$

$$= \frac{\frac{9}{13} - 3}{2}$$

$$= \frac{9 - 39}{26}$$

$$= \frac{-30}{26}$$

$$= \frac{-15}{13}$$

Required equation of circle

$$(x + 3)^2 + (y + 1)^2 = 4$$

$$\left(x + \frac{15}{13}\right)^2 + \left(y - \frac{3}{13}\right)^2 = 4 \quad \text{Ans.}$$

(c) A(5, 1) and tangent to the line $2x - y - 10 = 0$ at B (3, – 4)

Solution:

We know that General Equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ (I)

For A (5, 1) $(5)^2 + (1)^2 + 2g(5) + 2f(1) + c = 0$

$25 + 1 + 10g + 2f + c = 0$

$26 + 10g + 2f + c = 0$ (i)

For B (3, – 4) $(3)^2 + (-4)^2 + 2g(3) + 2f(-4) + c = 0$

$9 + 16 + 6g - 8f + c = 0$

$25 + 6g - 8f + c = 0$ (ii)

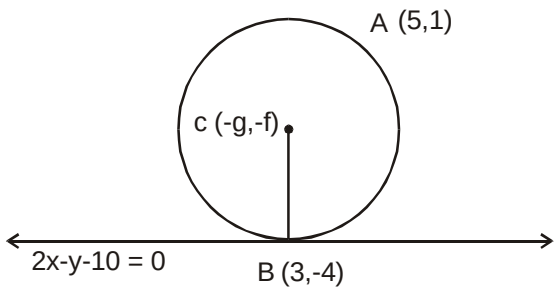
Subtracting (ii) from (i)

$$\begin{array}{rl} & 26 + 10g + 2f + c = 0 \\ - & 25 \pm 6g \mp 8f \pm c = 0 \\ \hline & 1 + 4g + 10f = 0 \quad \text{(iii)} \end{array}$$

m_1 = Slope of BC $= \dfrac{-f+4}{-g-3} = \dfrac{+(f-4)}{+(g+3)} = \dfrac{f-4}{g+3}$

m_2 = Slope of line $2x - y - 10 = 0$ is $\dfrac{-\text{coefficient of } x}{\text{coefficient of } y}$

$= -\dfrac{2}{-1} = 2$



Since lines are perpendicular

So $m_1 m_2 = -1$

$\dfrac{f-4}{g+3} \times 2 = -1$

$2f - 8 = -g - 3$

$-5 + g + 2f = 0$ (iv)

Now solving (iii) and (iv)

Multiply equation (iv) by 4 and subtracting from (iii)

$$\begin{array}{rl} 1 + 4g + 10f & = 0 \\ \mp 20 \pm 4g \pm 8f & = 0 \\ \hline 21 + 2f & = 0 \end{array}$$

$\boxed{f = -\dfrac{21}{2}}$ Put in (iv)

$-5 + g + 2\left(\dfrac{-21}{2}\right) = 0$

$-26 + g = 0$ $\boxed{g = 26}$ Putting values of g & f in (i)

$26 + 10(26) + 2\left(\dfrac{-21}{2}\right) + c = 0$

$26 + 260 - 21 + c = 0$

$\boxed{c = -265}$

Substitute all values in (I)

$$x^2 + y^2 + 2(26)x + 2\left(\frac{-21}{2}\right)y - 265 = 0$$

$$x^2 + y^2 + 52x - 21y - 265 = 0 \quad \text{Ans}$$

(d) A (1, 4), B (– 1, 8) and tangent to the line x + 3y – 3= 0

Solution:

We know that standard form of an equation of circle is $(x - h)^2 + (y - k)^2 = r^2$ (I)

For A (1, 4) $\quad (1 - h)^2 + (4 - k)^2 = r^2 \quad$ (i)

For B (–1, 8) $\quad -(-1 - h)^2 \pm (8 - k)^2 = -r^2 \quad$ (ii)

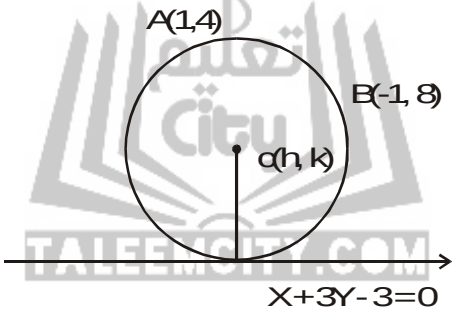
Subtracting $\quad (1 - h)^2 - (-1 - h)^2 + (4 - k)^2 - (8 - k)^2 = 0$

$$1 + h^2 - 2h - 1 - h^2 - 2h + 16 + k^2 - 8k - 64 - k^2 + 16k = 0$$

$$-4h + 8k - 48 = 0 \quad \Rightarrow \quad -4(h - 2k + 12) = 0$$

$$\Rightarrow \quad h - 2k + 12 = 0$$

$$h = 2k - 12 \quad \text{(iii)}$$



Since Circle (I) is tanget to the line $x + 3y - 3 = 0$

By perpendicular distance formula

$$r = \frac{|1(h) + 3k - 3|}{\sqrt{(1)^2 + (3)^2}}$$

$$r = \frac{h + 3k - 3}{\sqrt{10}} \quad \text{squaring} \quad r^2 = \frac{(h + 3k - 3)^2}{10} \quad \text{(A)}$$

$$10r^2 = h^2 + 9k^2 + 9 + 6hk - 18k - 6h$$

Using (ii)

$$10[1 + h^2 - 2h + 16 + k^2 - 8k] = h^2 + 9k^2 + 9 + 6hk - 18k - 6h$$

$$10 + 10h^2 - 20h + 160 + 10k^2 - 80k - h^2 - 9k^2 - 9 - 6nk + 18k + 6h = 0$$

$$9h^2 + k^2 - 14h - 62k - 64k + 161 = 0 \quad \text{(iv)}$$

Using (iii) in (iv) we have

$$9(2k - 12)^2 + k^2 - 14(2k - 12) - 62k - 6k(2k - 12) + 161 = 0$$

$$36k^2 + 1296 - 432k + k^2 - 28k + 168 - 62k - 12k^2 + 72k + 161 = 0$$
$$25k^2 - 450k + 1625 = 0$$
$$25(k^2 - 18k + 65) = 0$$
$$k^2 - 18k + 65 = 0$$
$$(k - 5)(k - 13) = 0$$
$$k = 5 \qquad k = 13$$

If $k = 5$
$h = 2(5) - 12$
$= 10 - 12$
$= -2$

If $k = 13$
$h = 2(13) - 12$
$= 26 - 12$
$= 14$

Putting values of h and k in "A"

$$r^2 = \left(\frac{-2 + 15 - 3}{10}\right)^2 \qquad r^2 = \left(\frac{14 + 39 - 3}{10}\right)^2$$

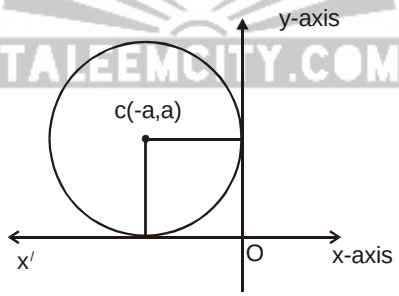
$$r^2 = \frac{(10)^2}{10} = 10 \qquad r^2 = \frac{(50)^2}{10} = \frac{2500}{10}$$

Required equations of circle are

$(x + 2)^2 + (y - 5)^2 = 10$ and $(x - 14)^2 + (y - 13)^2 = 250$ Ans

Q.5: Find an equation of a circle of radius a and lying in the second quadrant such that it is tangent to both the axes.

Solution:



As the circle is in 2nd quadrant and it is tangent to both the axes.

Therefore its center is (–a, a) and radius is a equation of circle by standard form is

$(x - h)^2 + (y - k)^2 = r^2$

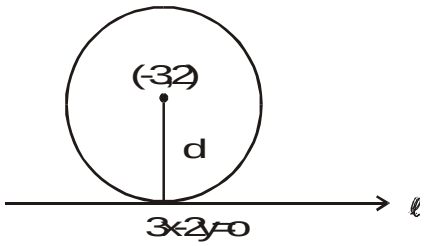
i.e; $(x + a)^2 + (y - a)^2 = a^2$

$x^2 + a^2 + 2ax + y^2 + a^2 - 2ay - a^2 = 0$

$x^2 + y^2 + 2ax - 2ay + a^2 = 0$ Ans

Q.6: Show that the lines $3x - 2y = 0$ and $2x + 3y - 13 = 0$ are tangents to the circle $x^2 + y^2 + 6x - 4y = 0$.

Solution:



Given circle is $x^2 + y^2 + 6x - 4y = 0$

Center is $(-g, -f) = (-3, 2)$ $\quad (\because 2g = 6,\ 2f = -4)$

Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{9 + 4} = \sqrt{13}$

The given two lines will be tangent to the given circle, if we find distance from the center of the circle to the given lines and it will equal to the radius of circle.

(i) Now bg perpendicular distance formula for line $3x - 2y = 0$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|3(-3) - 2(2)|}{\sqrt{(3)^2 + (-2)^2}}$$

$$= \frac{|-9 - 4|}{\sqrt{9 + 4}}$$

$$= \frac{|-13|}{\sqrt{13}}$$

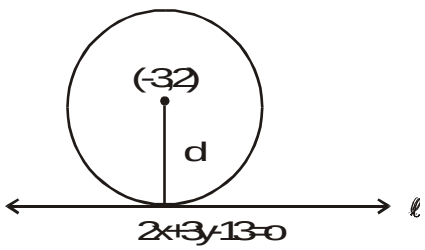
$$= \frac{13}{\sqrt{13}} = \sqrt{13}$$

Which show that $\boxed{d = \text{radius}}$

Hence $3x - 2y = 0$ is tangent to circle.

(ii) Now we will take the second line

$2x + 3y - 13 = 0$



$$d = \frac{|2(-3) + 3(2) - 13|}{\sqrt{(2)^2 + (3)^2}}$$

$$= \frac{|-6 + 6 - 13|}{\sqrt{13}}$$

$$= \frac{|-13|}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

Which is equal to radius $2x + 3y - 13 = 0$ is also tangent to given circle.

Q.7: Show that the circles $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$ touch externally.

Solution:

Given circles are

$x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$

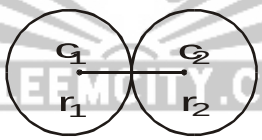
Center $= C_1 = (-1, 1)$ center $= C_2 = (3, -2)$

Radius $= r_1 = \sqrt{1 + 1 + 7} = \sqrt{9} = 3$ Radius $= r_2 = \sqrt{9 + 4 + 9} = \sqrt{4} = 2$

The two circles will touch each other externally if $r_1 + r_2 = |C_1 C_2|$

Note:

The two circle touch externally if their centers distance is equal to sum of their radius.



$$3 + 2 = \sqrt{(3 + 1)^2 + (-2 - 1)^2}$$

$$5 = \sqrt{16 + 9} \quad \Rightarrow \quad 5 = 5$$ Hence proved.

Q.8: Show that the circles $x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touch internally.

Solution:

Given, circles are

$x^2 + y^2 + 2x + 0y - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$

Center $C_1 = (-1, 0)$

Radius $= r_1 = \sqrt{1 + 0 - (-8)}$

$= \sqrt{1 + 8}$

$= \sqrt{9}$

$r_1 = 3$

Center $= C_2 = (3, -3)$

Radius $= r_2 = \sqrt{9 + 9 - (-46)}$

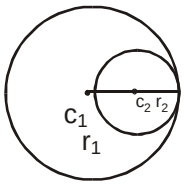
$= \sqrt{9 + 9 + 46}$

$= \sqrt{64}$

$r_2 = 8$

Recall

Two circles touch internally if their centers distance is equal to difference of their radii.



The two circles will touch internally

if $r_2 - r_1 = |C_1 C_2|$

$$8 - 3 = \sqrt{(3+1)^2 + (-3-0)^2}$$

$$5 = \sqrt{16+9}$$

$$5 = \sqrt{25}$$

$$5 = 5$$

Hence proved.

Q.9: Find equation of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at A(1, – 3).

Solution: Let (h, k) be center of circle with radius 2. Equation of circle by standard form is $(x-h)^2 + (y-k)^2 = r^2$ (I)

For A(1, – 3) and r = 2 (I) becomes

$$(1-h)^2 + (-3-k)^2 = 4$$

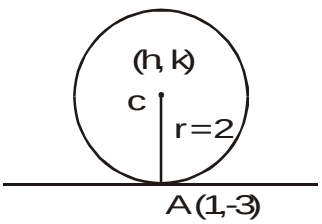
$$1 + h^2 - 2h + 9 + k^2 + 6k - 4 = 0$$

$$h^2 + k^2 - 2h + 6k + 6 = 0 \quad \text{(i)}$$

Slope of line $x - y - 4 = 0$ is

$$m_1 = \frac{-\text{coefficient of } x}{\text{coefficient of } y} = -\frac{1}{-1}$$

$$m_1 = 1$$



Slope of AC $= m_2 = \dfrac{k+3}{h-1}$ $\left(\because \dfrac{y_2 - y_1}{x_2 - x_1}\right)$

Since lines are perpendicular so

$$m_1 m_2 = -1$$

$$1 \times \frac{k+3}{h-1} = -1$$

$$k + 3 = -h + 1$$

$$k = -h - 2 \qquad \text{Put in (i)}$$

$$h^2 + (-h-2)^2 - 2h + 6(-h-2) + 6 = 0$$

$$h^2 + h^2 + 4 + 4h - 2h - 6h - 12 + 6 = 0$$

$$2h^2 - 4h - 2 = 0$$

$$h^2 - 2h - 1 = 0$$

$$h = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

$$= \frac{2[1 \pm \sqrt{2}]}{2}$$

$$h = 1 \pm \sqrt{2}$$

At $h = 1 + \sqrt{2}$

$k = -1 - \sqrt{2} - 2$

$k = -3 - \sqrt{2}$

At $h = 1 - \sqrt{2}$

At $k = -1 + \sqrt{2} - \sqrt{2}$

$k = -3 + \sqrt{2}$

Equations of circles are

$$(x - 1 - \sqrt{2})^2 + (y + 3 - \sqrt{2})^2 = 4 \quad \& \quad (x - 1 + \sqrt{2})^2 + (y + 3 - \sqrt{2})^2 = 4$$

Tangents

A tangent to a curve is a line that touches the curve without cutting through it.

Let $y = f(x)$ or $f(x, y) = 0$

$\frac{dy}{dx}$ is slope of tangent at any point P (x, y) to the curve.

Normal

The perpendicular to the tangent line is called normal to the curve.

Note

Slope of normal is negative reciprocal of slope of tangent.