EXERCISE 6.4

Q.1: Find the focus, vertex and directrix of the parabola. Sketch its graph.

(i) $y^2 = 8x$ **(Gujranwala Board 2007)**

Solution:

$y^2 = 8x$

As standard form is

$y^2 = 4ax$

$4a = 8 \Rightarrow \boxed{a = 2}$

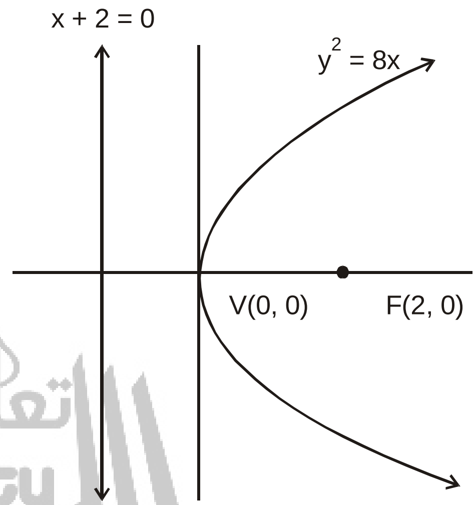
Focus $= (a, 0) = (2, 0)$

Vertex $= (0, 0) = (0, 0)$

Directrix $x = -a$

$x = -2$

$\Rightarrow x + 2 = 0$



(ii) $x^2 = -16y$

Solution:

$x^2 = -16y$

As standard form is

$x^2 = -4ay$

$\Rightarrow -4a = -16 \Rightarrow$

$\boxed{a = +4}$

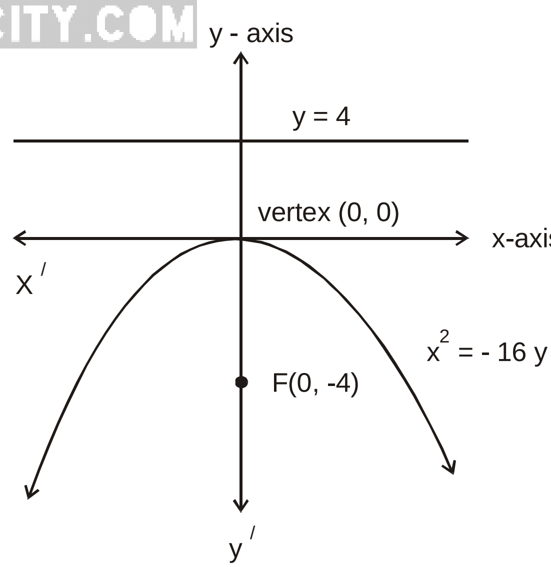
Focus $= (0, -a)$

Focus $= (0, -4)$

Vertex $= (0, 0)$

Directrix $y = a$

$y = 4$



(iii) $\mathbf{x^2 = 5y}$

Solution:

$x^2 = 5y$

As standard form is

$x^2 = 4ay$

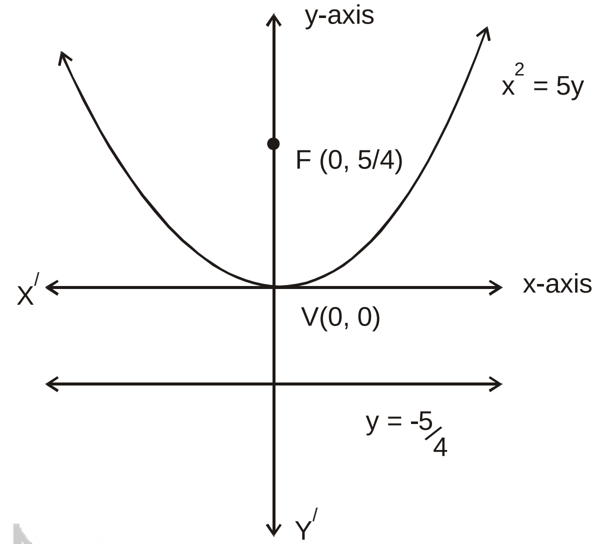
$4a = 5 \quad => \quad \boxed{a = \frac{5}{4}}$

Focus $= (0, a) = (0, \frac{5}{4})$

Vertex $= (0, 0)$

Directrix $y = -a$

$y = -\frac{5}{4}$



(iv) $\mathbf{y^2 = -12x}$

Solution:

$y^2 = -12x$

As standard form is

$y^2 = -4ax$

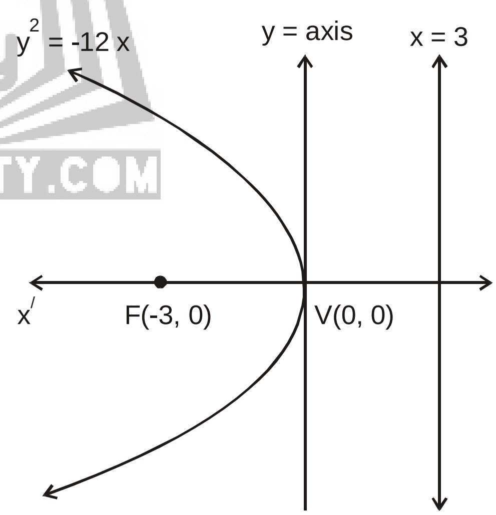
$4a = 12 \quad => \quad \boxed{a = 3}$

Focus $= (-a, 0) = (-3, 0)$

Vertex $= (0, 0)$

Directrix $x = a$

$x = 3$



(v) $\mathbf{x^2 = 4(y-1)}$

Solution:

$x^2 = 4(y-1)$

$(x-0)^2 = 4(y-1) \quad .... (1)$

Let $x - 0 = X, \quad y - 1 = Y$

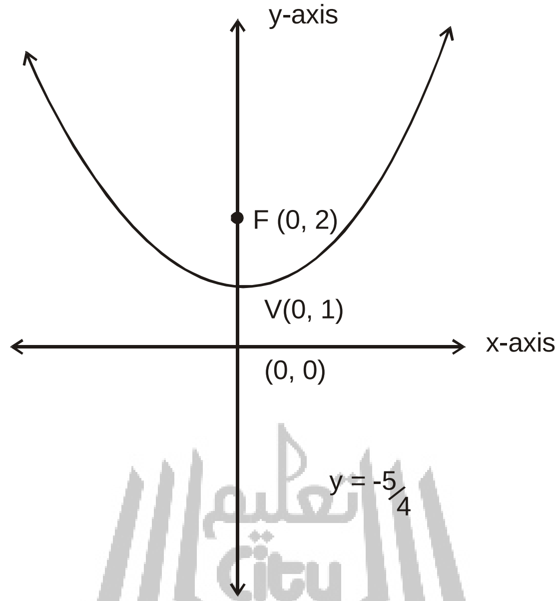
(1) Becomes

$x^2 = 4Y$

As standard form is

$x^2 = 4ay$

$4a = 4$ $\boxed{a = 1}$



Focus $= (0, a)$

$(X, Y) = (0,1)$

$(x, y-1) = (0,1)$

$x = 0 \quad y - 1 = 1$

$x = 0 \quad y = 2$

Focus $= (0, 2)$

For the vertex put $X = 0,\ Y = 0$

$x - 0 = 0 \quad , \quad y - 1 = 0$

$x = 0 \quad , \quad y = 1$

Vertex $= (0, 1)$

Directrix $Y = -a$

$y - 1 = -1$

$y = -1 + 1$

$\boxed{y = 0}$

(vi) $y^2 = -8(x-3)$

Solution:

$y^2 = -8(x-3)$

$(y-0)^2 = -8(x-2)$ .... (1)

Let $y-0 = Y$, $x-3 = X$

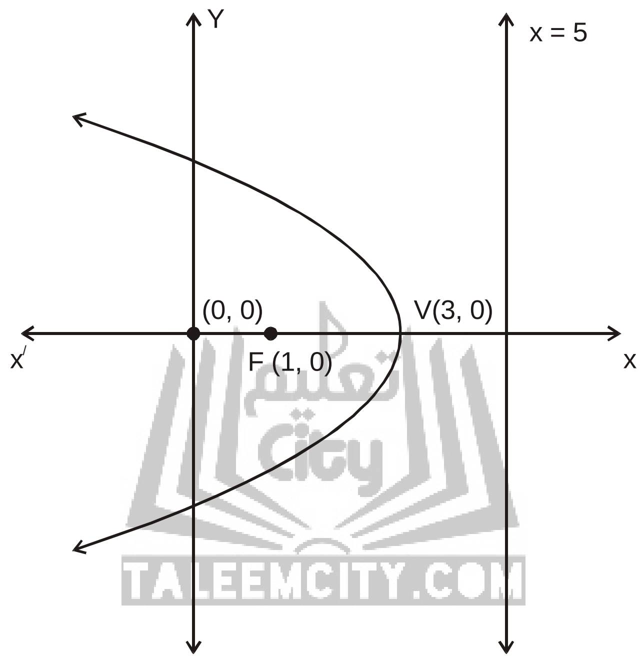
(1) Becomes

$Y^2 = -8X$

As standard form is

$y^2 = -4ax$

$4a = 8$ $\boxed{a = 2}$



Focus $= (-a, 0)$

$(X, Y) = (-2, 0)$

$(x-3, y-0) = (-2, 0)$

$x-3 = -2$, $y-0 = 0$

$x = -2+3$ $y = 0$

$x = 1$, $y = 0$

Focus $= (1, 0)$

For the vertex

Put $X = 0$, $Y = 0$

$x-3 = 0$, $y - 0 = 0$

$x = 3$, $y = 0$

Vertex $= (3, 0)$

Directrix $X = a$

$x - 3 = 2$

$x = 5$

(vii) $(x - 1)^2 = 8(y + 2)$ (Lahore Board 2009)

Solution:

$(x - 1)^2 = 8(y + 2)$ (i)

Let $x - 1 = X$, $y + 2 = Y$

(i) Becomes $X^2 = 8Y$

As standard form is

$x^2 = 4ay$

$4a = 8$

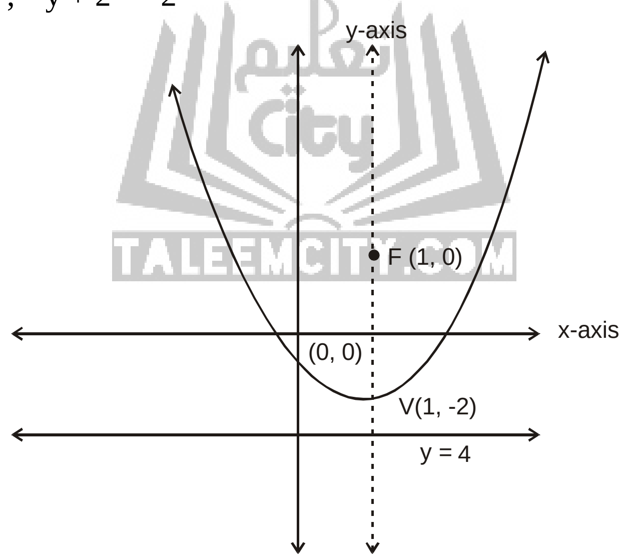
$\boxed{a = 2}$

Focus $= (0, a)$

$(X, Y) = (0, 2)$

$(x - 1, y + 2) = (0, 2)$

$x - 1 = 0$, $y + 2 = 2$



$x = 1$, $y = 2 - 2$

$x = 1$, $y = 0$

$\boxed{\text{Focus} = (1,0)}$

For the vertex put $X = 0$, $Y = 0$

$x - 1 = 0$, $y + 2 = 0$

$x = 1$, $y = -2$

$\boxed{\text{Vertex} = (1, -2)}$

directrix $Y = a$

$y + 2 = -2$

$y = -2 - 2$

$\boxed{y = -4}$

(viii) $\mathbf{y = 6x^2 - 1}$

Solution:

$y = 6x^2 - 1$

$6x^2 = y + 1$

$x^2 = \frac{1}{6}(y + 1)$

$\Rightarrow (x - 0)^2 = \frac{1}{6}(y + 1)$ .... (i)

Let $x - 0 = X$ $\quad y + 1 = Y$

(i) Becomes $X^2 = \frac{1}{6}y$

$x^2 = 4ay$

As standard form is

$4a = \frac{1}{6} \Rightarrow \boxed{a = \frac{1}{24}}$

Focus $= (0, a)$

$(X, Y) = (0, \frac{1}{24})$

$(x - 0 \quad , \quad y + 1) = (0, \frac{1}{24})$

$x - 0 = 0 \quad , \quad y + 1 = \frac{1}{24}$

$x = 0 \quad , \quad y = \frac{1}{24} - 1$

$y = \frac{-23}{24}$

$F = (0, \frac{-23}{24})$

For the vertex

Put $X = 0 \quad , \quad Y = 0$

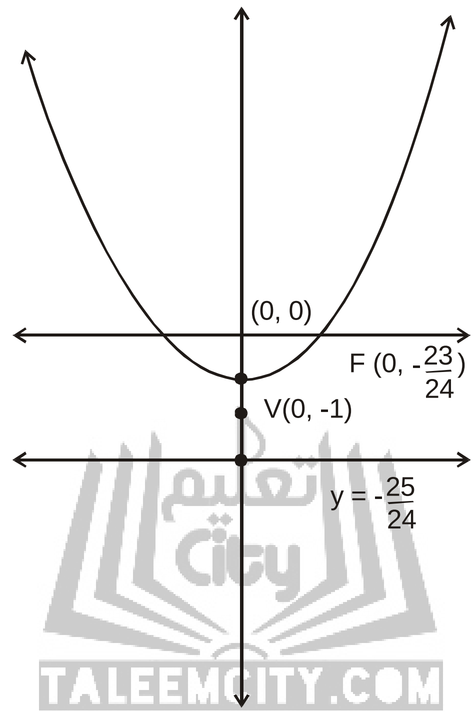
$x - 0 = 0 \quad , \quad y + 1 = 0$

$x = 0 \quad , \quad y = -1$

$\boxed{\text{Vertex} = (0, -1)}$

Directrix $Y = -a$

$y + 1 = -\dfrac{1}{24}$ $\quad y = -\dfrac{1}{24} - 1 = \boxed{\dfrac{-25}{24} = y}$



(ix) $x + 8 - y^2 + 2y = 0$ (Lahore Board 2011)

Solution:

$x + 8 - y^2 + 2y = 0$

$y^2 - 2y = x + 8$

$y^2 - 2y + 1 = x + 8 + 1$

$(y - 1)^2 = x + 9 \qquad \text{(i)}$

Let $y - 1 = Y$, $x + 9 = X$

(i) becomes

$Y^2 = X$

As standard form is

$y^2 = 4ax$

$4a = 1$

$\boxed{a = \dfrac{1}{4}}$

$$\text{Focus} = (a, 0)$$
$$(X, Y) = \left(\frac{1}{4}, 0\right)$$
$$(x + 9,\ \ y - 1) = \left(\frac{1}{4}, 0\right)$$
$$x + 9 = \frac{1}{4}, \quad y - 1 = 0$$
$$x = \frac{1}{4} - 9 \quad y = 1$$
$$x = \frac{-35}{4}$$
$$\boxed{\text{Focus} \left(\frac{-35}{4},\ 1\right)}$$

For the vertex put $X = 0$, $Y = 0$

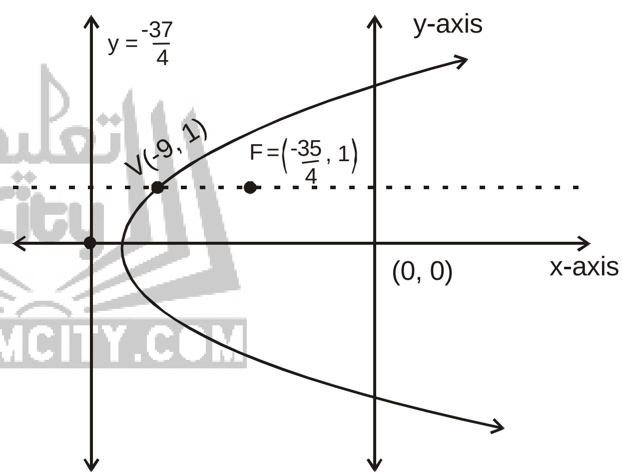
$x + 9 = 0$, $y - 1 = 0$

$x = -9$, $y = 1$

Required $\boxed{\text{Vertex} = (-9, 1)}$

Directrix is

$$X = -a$$
$$x + 9 = -\frac{1}{4}$$
$$x = -9 - \frac{1}{4}$$
$$\boxed{x = \frac{-37}{4}}$$



(x) $\mathbf{x^2 - 4x - 8y + 4 = 0}$ **(Lahore Board 2011)**

Solution:

$$x^2 - 4x = 8y - 4 = 0$$
$$x^2 - 4x + 4 = 8y - 4 + 4$$
$$(x - 2)^2 = 8y \quad \text{..... (i)}$$

Let $x - 2 = X$ $y - 0 = Y$

(i) becomes

$$X^2 = 8Y$$

As standard form is

$x^2 = 4ay$

$4a = 8$

$\boxed{a = 2}$

Focus $= (0, a)$

$(X, Y) = (0\ 2)$

$(x - 2, y - 0) = (0, 2)$

$x - 2 = 0 \quad , \quad y - 0 = 2$

$x = 2 \quad , \quad y = 2$

$\boxed{\text{Focus} = (2, 2)}$

For the vertex

Put $X = 0 \quad , \quad Y = 0$

$(X, Y) = (0, 0)$

$(x - 2, \ y - 0) = (0, 0)$

$x - 2 = 0 \quad , \quad y - 0 = 0$

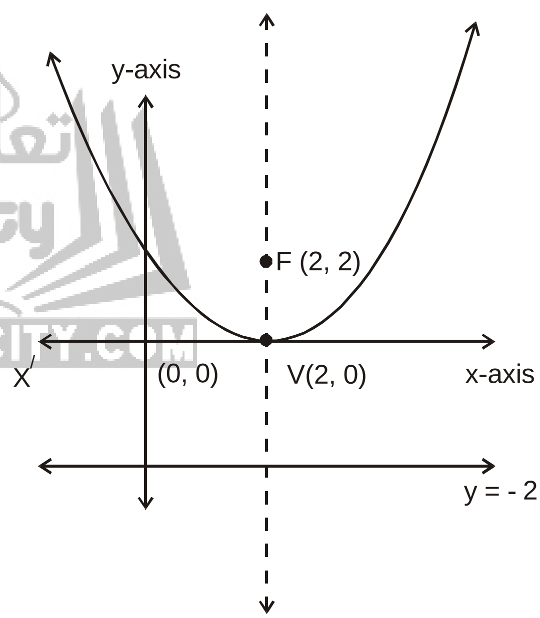
$x = 2 \quad , \quad y = 0$

$\boxed{\text{Vertex} = (2, 0)}$

Directrix $Y = -a$

$y - 0 = -2$

$\boxed{y = -2}$



Q.2: Write an equation of the Parabola with given elements.

(i) Focus $(-3, 1)$; Directrix $x = 3$

Solution:

Given $F = (-3, 1)$

& directrix $x + 0y - 3 = 0$

Let $P(x, y)$ be any point on the Parabola. Then $|PM|$ = Length of Perpendicular from $P(x, y)$ to the directrix L.

$$|PM| = \frac{|x + 0y - 3|}{\sqrt{(1)^2 + (0)^2}}$$

By definition of Parabola

$|PF| = |PM|$

or $|PF|^2 = |PM|^2$

=> $(x + 3)^2 + (y - 1)^2 = (x + 0y - 3)^2$

$x^2 + 9 + 6x + y^2 + 1 - 2y = x^2 + 9 - 6x$

$y^2 - 2y + 1 = -12x$

$(y - 1)^2 = -12x$ Ans.

(ii) Focus (2, 5) ; directrix y = 1

Solution:

Given $F = (2, 5)$

directrix $0x + y - 1 = 0$

Let P(x, y) be any point on the Parabola.

Then |PM| = Length of Perpendicular from P(x, y) to directrix

$$|PM| = \frac{|0x + y - 1|}{\sqrt{(0)^2 + (1)^2}} = y - 1$$

Now, by definition of Parabola

$|PF| = |PM|$

=> $|PF|^2 = |PM|^2$

$(x - 2)^2 + (y - 5)^2 = (y - 1)^2$

$x^2 + 4 - 4x + y^2 + 25 - 10y = y^2 + 1 - 2y$

$x^2 - 4x - 8y + 28 = 0$ Ans

(iii) Focus (–3, 1) ; directrix x – 2y – 3 = 0

Solution:

Given Focus $(-3, 1)$

directrix $x - 2y - 3 = 0$

Let P(x, y) be any point on the Parabola

Then |PM| = distance or length of Perpendicular from P(x, y) to the directrix.

$$|PM| = \frac{|x - 2y - 3|}{\sqrt{(1)^2 + (-2)^2}} = \frac{(x - 2y - 3)}{\sqrt{5}}$$

By definition of Parabola

$|PF| = |PM|$ => $|PF|^2 = |PM|^2$

=> $$(x + 3)^2 + (y - 1)^2 = \frac{(x - 2y - 3)^2}{5}$$

$5[x^2 + 9 + 6x + y^2 + 1 - 2y] = x^2 + 4y^2 + 9 - 4xy + 12y - 6x$

$5x^2 + 45 + 30x + 5y^2 - 10y - x^2 - 4y^2 - 9 + 4xy - 12y + 6x = 0$

$4x^2 + y^2 + 4xy + 36x - 22y + 36 = 0$ Ans.

(iv) Focus (1, 2); Vertex (3, 2)

Solution:

Given Focus = (1, 2) , Vertex = (3, 2)

We know that a = distance between focus & vertex

$a = \sqrt{(3-1)^2 + (2-2)^2} = \sqrt{4+0} = 2$

Required equation of Parabola

$(y - k)^2 = -4a(x - h)$

$(y - 2)^2 = -4(2)(x - 3)$

$y^2 + 4 - 4y = -8x + 24$

$y^2 - 4y + 8x - 20 = 0$ Ans.

(v) Focus (– 1 , 0) ; Vertex (– 1, 2)

Solution:

F = (– 1, 0) , V = (– 1, 2)

a = distance between focus to vertex

$= \sqrt{(-1+1)^2 + (2-0)^2} = 2$

Required equation of Parabola is

$(x - h)^2 = -4a(y - k)$

$(x + 1)^2 = -4(2)(y - 2)$

$x^2 + 1 + 2x = -8y + 16$

$x^2 + 2x + 8y - 15 = 0$ Ans.

(vi) Directrix x = – 2 ; Focus (2, 2)

Solution:

Given F = (2, 2) directrix $x + 0y + 2 = 0$

$|PM|$ = distance or length of perpendicular from p (x, y) to the directrix.

$|PM| = \dfrac{|x + 0y + 2|}{\sqrt{1^2 + 0^2}} = x + 2$

By definition of Parabola

$|PF| = |PM|$

$\Rightarrow |PF|^2 = |PM|^2$

$\Rightarrow (x - 2)^2 + (y - 2)^2 = (x + 2)^2$

$x^2 + 4 - 4x + y^2 + 4 - 4y = x^2 + 4 + 4x$

$y^2 - 4y - 8x + 4 = 0$ Ans

(vii) Directrix $y = 3$; Vertex $(2, 2)$

Solution:

Directrix $ox + y - 3 = 0$ $V = (2, 2)$

We know that a = distance between directrix and vertex

$$= a = \frac{|0(2) + 1(2) - 3|}{\sqrt{0^2 + 1^2}} = \frac{|2 - 3|}{\sqrt{1}} = |-1| = 1$$

Since the directrix is above the vertex,

Therefore equation of Parabola is $(x - h)^2 = -49\,(y - k)$

$(x - 2)^2 = -4(1)\,(y - 2)$

$x^2 + 4 - 4x = -4y + 8$

$x^2 + 4 - 4x + 4y - 8 = 0$

$x^2 - 4x + 4y - 4 = 0$

(viii) Directrix y = 1, Length of latusrectum is 8.0 and opens downward.

Solution:

Given $4a = 8$ $\boxed{a = 2}$

As Parabola opens downward, so its equation is of the form

$(x - h)^2 = -4a(y - k)$ .... (1)

We know that vertex is below the directrix y = 1

So y – coordinate of the vertex is = y + a

$1 = y + 2 \quad \Rightarrow \quad y = -1 \quad$ i.e; $\boxed{k = -1}$

with a = 2 & k = – 1 equation (1) becomes

$$(x - h)^2 = -4(2)(y + 1)$$
$$x^2 + h^2 - 2hx = -8y - 8$$
$$x^2 + h^2 - 2hx + 8y + 8 = 0 \qquad \text{Ans.}$$

(ix) Axis y = 0, through (2, 1) & (11, – 2)

Solution:

As axis y = 0, so required equation of the parabola is $(y - k)^2 = 4a(x - h)$ (1) because of the axis of Parabola is x-axis & y = 0 so k = 0

$\therefore$ with k = 0 equation (1) becomes

$y^2 = 4a(x - h)$ .... (2)

Since the para-bola passes through the points (2, 1) & (11, – 2) equation (2) becomes

For (2, 1) For (11, – 2)

$1 = 4a(2 - h)$ $\qquad 4 = 4a(11 - h)$

$1 = 8a - 4ah$ (3) $\qquad 4 = 44a - 4ah$ (4)

Subtracting (3) from (4) we have

$$\begin{array}{rcl} 4 = & 44a & - \ 4ah \\ -\ 1 = - & 8a & \mp \ 4ah \\ \hline 3 = & 36a & \end{array}$$

$$\boxed{a = \frac{1}{12}}$$

Put in (3)

$$1 = 8\left(\frac{1}{12}\right) - 4\left(\frac{1}{12}\right)h$$

$$1 = \frac{8}{12} - \frac{4}{12}h$$

$$1 = \frac{8-4h}{12} \quad \Rightarrow \quad 12 = 8 - 4h$$
$$\Rightarrow \quad 4 = -4h$$
$$\Rightarrow \quad \boxed{h = -1}$$

Equation (2) becomes

$$y^2 = 4 \times \frac{1}{12}(x+1)$$

$\boxed{3y^2 = x + 1}$ Ans.

(x) Axis parallel to y-axis. The points (0, 3) (3, 4) & (4, 11) lie on the graph.

Solution:

As axis of parabola parallel to y – axis, so its equation will be

$(x - h)^2 = 4a(y - k)$ .... (1)

As points (0, 3), (3, 4) and (4, 11) lies on the parabola (1) so $(0 - h)^2 = 4a(3 - k)$

$h^2 = 12a - 4ak$ .... (2)

For (3, 4) $\quad (3 - h)^2 = 4a(4 - k)$

$9 + h^2 - 6h = 16a - 4ak$ (3)

For (4, 11) $\quad (4 - h)^2 = 4a(11 - k)$

$16 + h^2 - 8h = 44a - 4ak$ (4)

Subtracting (2) from (3)

$$\begin{array}{rcl} 9 + h^2 - 6h &=& 16a - 4ak \\ -h^2 &=& -12a \mp 4ak \\ \hline 9 - 6h &=& 4a \quad (5) \end{array}$$

Subtracting (2) from (4)

$$\begin{array}{rcl} 16 + h^2 - 8h &=& 44a - 4ak \\ -h^2 &=& -12a \mp 4ak \\ \hline 16 - 8h &=& 32a \\ 8(2 - h) &=& 32a \\ 2 - h &=& 4a \quad (6) \end{array}$$

Subtracting (6) from (5)

$$\begin{array}{rcl} 9 - 6h &=& 4a \\ -2 \mp h &=& -4a \\ \hline 7 - 5h &=& 0 \end{array}$$

$$\Rightarrow \quad 7 = 5h$$

$\boxed{\frac{7}{5} = h}$ Put in 5

$$9 - 6h = 4a$$

$$9 - 6\left(\frac{7}{5}\right) = 4a$$

$$9 - \frac{42}{5} = 4a$$

$$\frac{45 - 42}{5} = 4a \quad => \quad \frac{3}{5} = 4a$$

$$\boxed{a = \frac{3}{20}}$$

Put in (2)

$$\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right) - 4\left(\frac{3}{20}\right)k$$

$$\frac{49}{25} = \frac{36}{20} - \frac{12}{20}k$$

$$\frac{49}{25} = \frac{36 - 12k}{20}$$

$$196 = 180 - 60k$$

$$60k = -16 \quad => \quad k = \frac{-16}{60} = \boxed{\frac{-4}{5} = k}$$

Substituting all values in (1)

$$\left(x - \frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y + \frac{4}{5}\right)$$

$$\left(x - \frac{7}{5}\right)^2 = \frac{3}{5}\left(y + \frac{4}{5}\right) \quad \text{Ans.}$$

Q.3: Find an equation of the Parabola having its focus at the origin and directrix Parallel to

(i) the x–axis

Solution:

Given F = (0, 0)

Directrix Parallel to x–axis

$0x + y - h = 0$

Let P(x, y) be any point on the Parabola such that

$$|PF| = |PM|$$

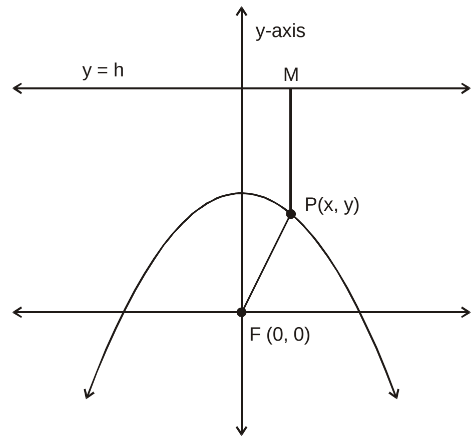
$$=> |PF|^2 = |PM|^2$$

$$(x-0)^2 + (y-0)^2 = \left(\frac{|0x + y - h|}{\sqrt{1^2 + 0^2}}\right)^2$$

$$x^2 + y^2 = (y-h)^2$$

$$x^2 + y^2 = y^2 + h^2 - 2yh$$

$x^2 + 2hy - h^2 = 0$ Required Equation



(ii) The y– axis.

Solution:

Given F = (0, 0)

Directrix Parall to y – axis

$x + 0y - h = 0$

Let P(x, y) be any point on the parabola such that

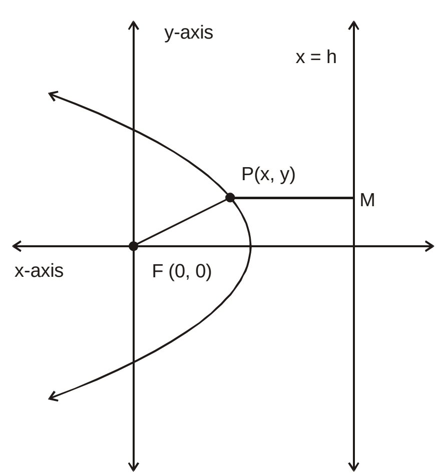
$|PF| = |PM|$

$\Rightarrow |PF|^2 = |PM|^2$

$(x - 0)^2 + (y - 0)^2 = (x + 0y - h)^2$

$x^2 + y^2 = x^2 + h^2 - 2xh$

$y^2 + 2xh - h^2 = 0$ required equation.



Q.4: Show that the parabola $(x \sin \alpha - y \cos \alpha)^2 = 4a (x \cos \alpha + y \sin \alpha)$ has focus at $(a \cos \alpha, a \sin \alpha)$ and its directrix is $x \cos \alpha + y \sin \alpha + a = 0$.

Solution:

Here Focus $= (a \cos \alpha, a \sin \alpha)$

directrix $M = x \cos \alpha + y \sin \alpha + a = 0$

Let P(x, y) be any point on the Parabola, such that

$|PF| = |PM|$

=> $|PF|^2 = |PM|^2$

$$(x - a \cos \alpha)^2 + (y - a \sin \alpha)^2 = \frac{(x \cos \alpha + y \sin \alpha + a)^2}{\sin 2\alpha + \cos 2\alpha}$$

$x^2 + a^2 \cos^2 \alpha - 2ax \cos \alpha + y^2 + a^2 \sin^2\alpha - 2ay \sin \alpha = x^2 \cos^2\alpha + y^2 \sin^2\alpha + a^2 + 2xy \sin \alpha \cos \alpha + 2ay \sin \alpha + 2ax \cos \alpha$

$x^2 - x^2 \cos^2 \alpha + y^2 - y^2 \sin^2\alpha + a^2 (\cos^2\alpha + \sin^2\alpha) = a^2 + 2xy \sin \alpha \cos \alpha + 2ay \sin \alpha + 2ax \cos \alpha + 2ay \sin \alpha + 2ax \cos \alpha$

$x^2 (1 - \cos^2\alpha) + y^2 (1 - \sin^2\alpha) + a^2 - 2xy \sin\alpha \cos\alpha = a^2 + 4ay \sin\alpha + 4ax \cos\alpha$

$x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy \sin \alpha \cos \alpha = 4ay \sin \alpha + 4ax \cos \alpha$

$(x \sin \alpha - y \cos \alpha)^2 = 4a(x \cos\alpha + y \sin \alpha)$ Hence proved.

Q.5: Show that the ordinate at any point P of the Parabola is mean Propotional between the length of the Latustrectum and the abscissa of P.

Solution:

Let P(x, y) be any point on the Parabola then the equation of Parabola is

$y^2 = 4ax$

$y.y = 4ax$

$$\frac{y}{4a} = \frac{x}{y}$$

=> $$\frac{x}{y} = \frac{y}{4a}$$

=> $x : y :: y : 4a$

=> abscissa : ordinate : : ordinate : Length of Latusrectum

Hence proved.

Q.6: A comet has a Parabolic orbit with the earth at the focus. When the comet is 150,000 km from the earth, the line joining the comet and the earth makes an angle of 30 with the axis of Parabola. How close will the comet come to the earth?

Solution:

Let focus be taken at origin, then F = (0, 0) and directrix $x + oy + 2a = 0$

Let P (x, y) be any point on the Parabola such that

$|PF| = |PM|$

$\Rightarrow$ $|PF|^2 = |PM|^2$

$(x - 0)^2 + (y - 0)^2 = (x + 0y + 2a)^2$

$x^2 + y^2 = (x + 2a)^2$ ...... (1)

Now, from right angled triangle EQP

By Pythagoras theorem

$(150000)^2 = x^2 + y^2$

Putting in (1) $(150000)^2 = (x + 2a)^2$

$x + 2a = 150000$ ...... (2)

& we know that $\cos\alpha = \dfrac{\text{base}}{\text{hypotenous}}$

$\cos 30^o = \dfrac{x}{150000}$ $\Rightarrow$ $x = \dfrac{\sqrt{3}\times 150000}{2} = \sqrt{3} \times 75000$

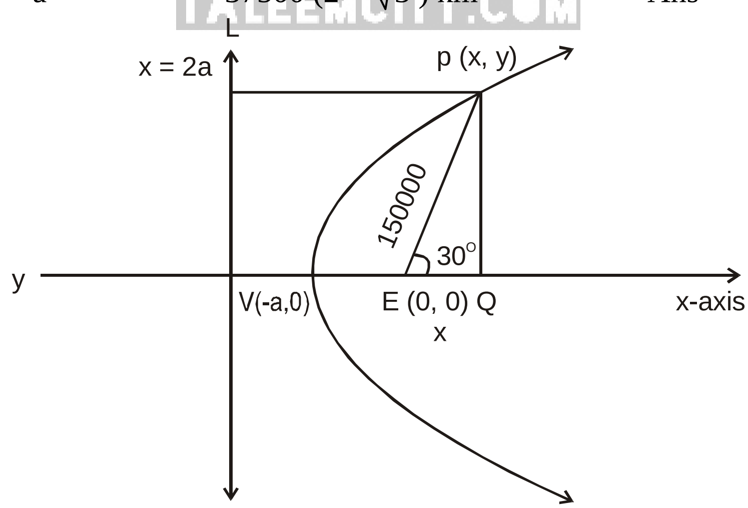
Put in (2)

$75000\sqrt{3} + 2a = 150000$

$2a = 150000 - 75000\sqrt{3}$

$a = \dfrac{75000}{2}(2-\sqrt{3})$

$a = 37500(2-\sqrt{3})$ km Ans



Q.7: Find an equation of the Parabola formed by the cables of a suspension bridge whose span is a m and the verticle height of the supporting towers is bm.

Solution:

We know that an equation of Parabolla is

$$x^2 = 4ay \qquad (1)$$

Since the point P $\left(\frac{a}{2}, b\right)$ lies on the parabola (1)

$\therefore$ (1) becomes

$$\left(\frac{a}{2}\right)^2 = 4a'b$$

$$\frac{a^2}{4} = 4a'b$$

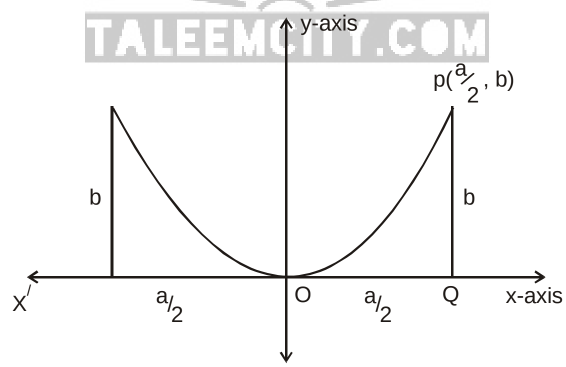
$$\frac{a^2}{4} = a'(4b)$$

$$a' = \frac{a^2}{4} \times \frac{1}{4b} = \frac{a^2}{16\,b}$$

Now putting value of a' in (1)

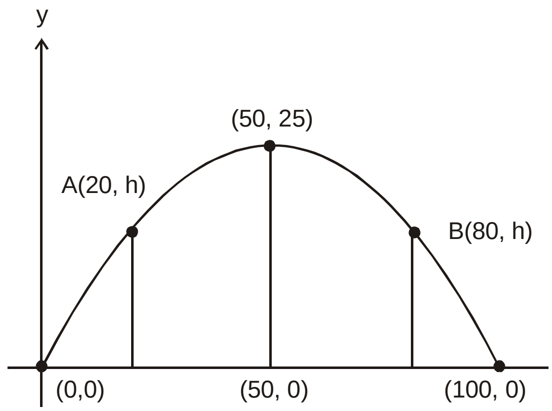
$$x^2 = 4\left(\frac{a^2}{16\,b}\right)y = \frac{a^2}{4b}\,y$$

$$x^2 = \frac{a^2 y}{4\,b} \qquad \text{Ans}$$



Q.8: A Parabolic arch has a 100 m base and height 25m. Find the height of the arch at the point 30 m from the center of the base.

Solution:



From the equation of Parabola is $(x - h)^2 = 4a(y - k)$ ..... (1)

With vertex V = (50, 25) (1) becomes

$(x - 50)^2 = 4a(y - 25)$ ..... (2)

Since origin 0(0, 0) lies on parabola

(2) becomes

$(0 - 50)^2 = 4a(0 - 25)$

$2500 = -100a$

$\boxed{a = -25}$

Putting value of a in (2) we have

$(x - 50)^2 = 4(-25)(y - 25)$ ..... (3)

Since point A(20, h) also lies on parabola

(3) becomes

$(20 - 50)^2 = -100(h - 25)$

$900 = -100h + 2500$

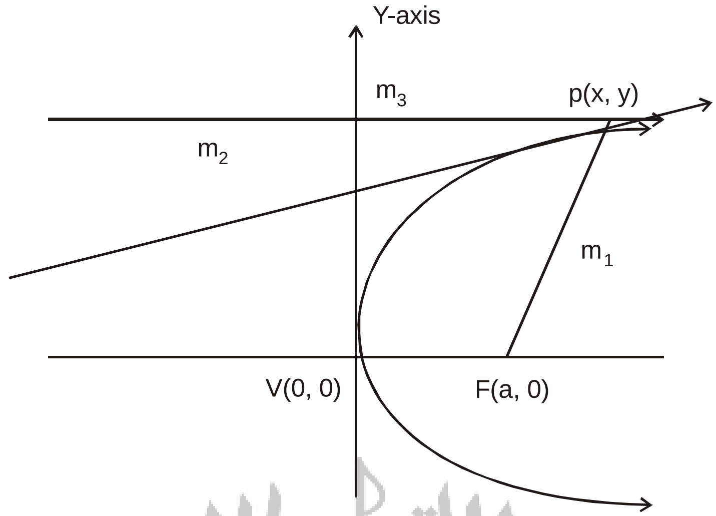
$100h = 2500 - 900$

$100h = 1600$

$h = 16$ m Ans

Q.9: Show that the tangent at any point P of a parabola makes equal angles with the line PF and the line through P Parallel to the axis of the Parabola, F being focus.

Solution:



We know that equation of parabola is

$$y^2 = 4ax \quad (1)$$

Diff. w.r.t 'x'

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$\therefore$ m_2 = Slope of tangent

$$= \frac{dy}{dx} = \frac{2a}{y}$$

m_1 = Slope of line between F and P

$$m_1 = \frac{y-0}{x-a} = \frac{y}{x-a}$$

we know that slope of x – axis = 0

$\therefore$ m_3 = Slope of line parallel to x – axis is also

Now $\tan\theta_1 = \dfrac{m_2 - m_3}{1 + m_2 m_3}$

$$\tan\theta_1 = \frac{\frac{2a}{y} - a}{1 + \left(\frac{2a}{y}\right)(0)}$$

$$= \frac{\frac{2a}{y}}{1} = \frac{2a}{y}$$

$$\theta_1 = \tan^{-1}\left(\frac{2a}{y}\right) \quad \text{..... (2)}$$

Next $\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$

$$= \frac{\frac{y}{x-a} - \frac{2a}{y}}{1 + \frac{y}{x-a} \times \frac{2a}{y}}$$

$$= \frac{y^2 - 2ax + 2a^2}{xy - ay + 2ay}$$

$$= \frac{y^2 - 2ax + 2a^2}{xy + ay} \quad \text{...... (3)}$$

Using (1)

$$\tan\theta = \frac{4ax - 2ax + 2a^2}{xy + ay} = \frac{2ax + 2a^2}{xy + ay} = \frac{2a(x+a)}{y(x+a)}$$

$$\tan\theta_2 = \frac{2a}{y}$$

$$\theta_2 = \tan^{-1}\frac{2a}{y} \quad \text{........ (4)}$$

From (2) & (4)

$\theta_1 = \theta_2$ Hence Proved.

Ellipse

Let 0 < 0 < 1 and F be a fixed point and L be a fixed line not containing F. Let P(x, y) be a point in the plane and |PM| be the perpendicular distance of P from L. The set of all the points P such that

$\frac{|PF|}{|PM|} = e$ is called an Ellipse.

e is eccentricity, F is focus, L is directrix.

Standard Form of or Ellipse (Lahore Board 2010)

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Special case of an Ellipse

Circle is a special case of an Ellipse. In circle ''e'' = 0

Parametric Equations of an Ellipse

$x = a\cos\theta$, $y = b\sin\theta$ are Parametric Equations of Ellipse.

Important points about an Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(1) Eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$(ae)^2 = a^2 - b^2$$

$$c^2 = a^2 - b^2$$

(2) Foci $(\pm ae, 0)$ or $(\pm c, 0)$

(3) Length of major axis = $2a$

(4) Length of minor axis = $2b$

(5) Equations of directrix x : $x = \pm \frac{a}{e}$

(6) Length of latus rectum = $\frac{2b^2}{a}$

(7) Center (0, 0)

(8) Vertices $(\pm a, 0)$

(9) Covertices $(0, \pm b)$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

(1) Eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$(ae)^2 = a^2 - b^2$$

$c^2 = a^2 - b^2$ where $c = ae$

(2) Foci $(0, \pm ae)$ or $(0, \pm c)$

(3) Length of major axis = $2a$

(4) Length of minor axis = $2b$

(5) Equations of directrix : $y = \pm \frac{a}{e}$

(6) Length of latus rectum = $\frac{2b^2}{a}$

(7) Center (0, 0)

(8) Vertices $(0, \pm a)$

(9) Covertices $(\pm b, 0)$

Note:

If center is other than (0, 0) then equations of ellipse be comes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

&

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

EXERCISE 6.5

Q.1: Find an equation of the Ellipse with given data and sketch its graph.

(i) Foci (±3, 0) and minor axis of length 10.

(Lahore Board 2009)

Solution: