(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Special case of an Ellipse

Circle is a special case of an Ellipse. In circle ''e'' = 0

Parametric Equations of an Ellipse

$x = a\cos\theta$, $y = b\sin\theta$ are Parametric Equations of Ellipse.

Important points about an Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(1) Eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$(ae)^2 = a^2 - b^2$$

$$c^2 = a^2 - b^2$$

(2) Foci $(\pm ae, 0)$ or $(\pm c, 0)$

(3) Length of major axis = $2a$

(4) Length of minor axis = $2b$

(5) Equations of directrix x : $x = \pm \frac{a}{e}$

(6) Length of latus rectum = $\frac{2b^2}{a}$

(7) Center (0, 0)

(8) Vertices $(\pm a, 0)$

(9) Covertices $(0, \pm b)$

Note:

If center is other than (0, 0) then equations of ellipse be comes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

(1) Eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$(ae)^2 = a^2 - b^2$$

$$c^2 = a^2 - b^2 \text{ where } c = ae$$

(2) Foci $(0, \pm ae)$ or $(0, \pm c)$

(3) Length of major axis = $2a$

(4) Length of minor axis = $2b$

(5) Equations of directrix : $y = \pm \frac{a}{e}$

(6) Length of latus rectum = $\frac{2b^2}{a}$

(7) Center (0, 0)

(8) Vertices $(0, \pm a)$

(9) Covertices $(\pm b, 0)$

& $$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

EXERCISE 6.5

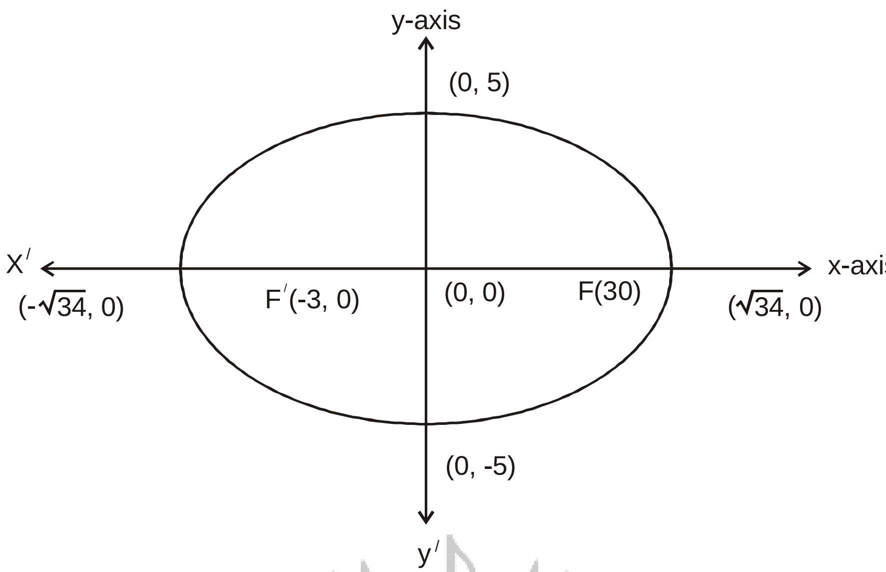
Q.1: Find an equation of the Ellipse with given data and sketch its graph.

(i) Foci (±3, 0) and minor axis of length 10.

(Lahore Board 2009)

Solution:

Given $(\pm ae, 0) = (\pm 3, 0)$



$ae = 3$

$\Rightarrow \quad c = 3$ $\qquad$ Also $2b = 10 \Rightarrow \boxed{b = 5}$

We know that $c^2 = a^2 - b^2$

$$(3)^2 = a^2 - (5)^2$$

$$9 = a^2 - 25$$

$$9 + 25 = a^2$$

$$a^2 = 34 \Rightarrow \boxed{a = \pm\sqrt{34}}$$

Required equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{x^2}{34} + \frac{y^2}{25} = 1$$

Here vertices are

$= (\pm a, 0)$

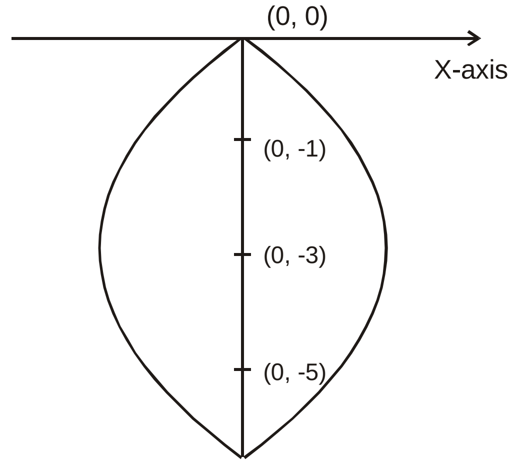
$= (\pm\sqrt{34}, 0)$

Covertices are $(0, \pm b)$

$= (0, \pm 5)$

(ii) Foci (0, – 1) & (0, – 5) and major axis of length 6.

Solution:



We know that center = mid point of foci

$$= \left(\frac{0+0}{2}, \frac{-1-5}{2}\right)$$

$$= (0, -3)$$

Also we know that

C = distance between centre and focus

$$C = \sqrt{(0-0)^2 + (-3+1)^2}$$

$$C = \sqrt{4} = 2$$

Given $2a = 6$

$a = 3$

We know for Ellipse

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad (2)^2 = (3)^2 - b^2$$

$$4 - 9 = -b^2 \quad \Rightarrow \quad b^2 = 5$$

With center (0, – 3), required equation of the ellipse is

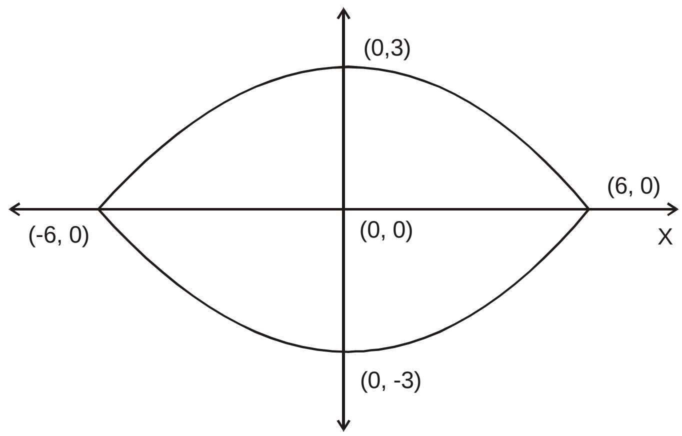
$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-0)^2}{5} + \frac{(y+3)^2}{(3)^2} = 1$$

$$\frac{x^2}{5} + \frac{(y+3)^2}{9} = 1$$

(iii) **Foci $(-3\sqrt{3}, 0)$ & Vertices $(\pm 6, 0)$**
(Lahore Board 2009)

Solution:



$(\pm ae, 0) = (\pm 3\sqrt{3}, 0)$ & $(\pm a, 0) = (\pm 6, 0)$

$ae = 3\sqrt{3}$ $\quad a = 6$

$c = 3\sqrt{3}$ $\quad a = 6$

We know that $c^2 = a^2 - b^2$

$(3\sqrt{3})^2 = (6)^2 - b^2$

$27 = 36 - b^2 \quad \Rightarrow \quad b^2 = 9$

Required equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

Covertices are $= (0, \pm b)$

$= (0, \pm 3)$

(iv) **Vertices (–1, 1), (5, 1)** **foci (4, 1) & (0, 1)**

Solution:

$2a = |VV'|$ $\qquad 2c = |FF'|$

$2a = \sqrt{(5+1)^2 + (1-1)^2}$ $\qquad 2c = \sqrt{(0+4)^2 + (1-1)^2}$

$2a = \sqrt{36}$ $\qquad 2c = \sqrt{16}$

$2a = 6$ $\qquad 2c = 4$

$\boxed{a = 3}$ $\qquad \boxed{c = 2}$

We know that $c^2 = a^2 - b^2$

$(2)^2 = (3)^2 - b^2$

$$4 = 9 - b^2$$

$$\boxed{b^2 = 5}$$

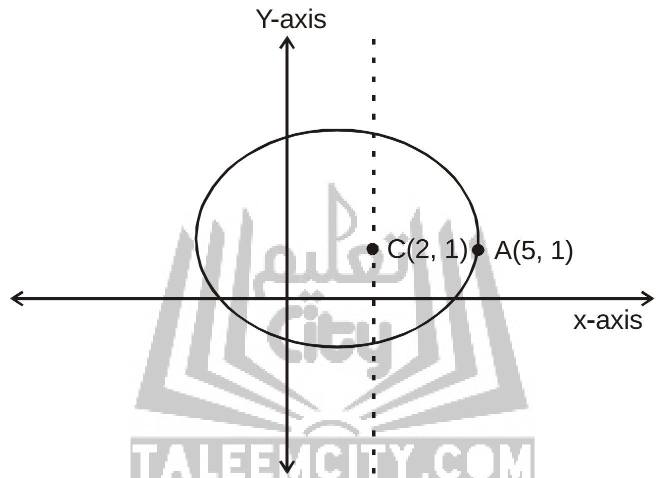
Now center of ellipse $= (h, k) =$ mid point of foci

$$= \left(\frac{4+0}{2}, \frac{1+1}{2}\right)$$

with this center required equation of ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

i.e; $$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1$$



(v) Foci $(\pm\sqrt{5}, 0)$ & passing through $\left(\frac{3}{2}, \sqrt{3}\right)$

Solution:

Given $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

$c = \sqrt{5}$ we know that $c^2 = a^2 - b^2$

$$(\sqrt{5})^2 = a^2 - b^2$$

$$5 = a^2 - b^2$$

$$\Rightarrow \quad a^2 = 5 + b^2 \quad \text{....... (1)}$$

Since Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and it is passing through point $\left(\frac{3}{2}, \sqrt{3}\right)$ therefore it becomes $\frac{9}{4a^2} + \frac{3}{b^2} = 1$ ......(2)

from (1) we have $a^2 = 5 + b^2$ Put in (2)

$$\frac{9}{4(5+b^2)} + \frac{3}{b^2} = 1$$

$$\frac{9b^2 + 12(5+b^2)}{4b^2(5+b^2)} = 1$$

$$9b^2 + 60 + 12b^2 = 20b^2 + 4b^4$$

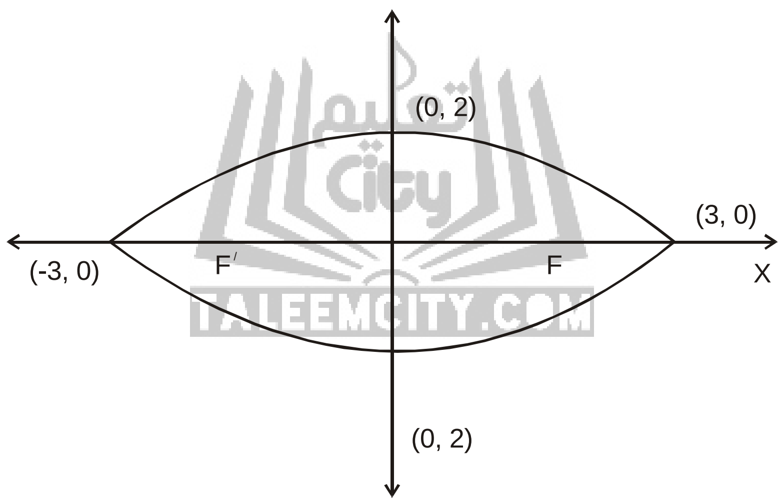
$$4b^4 - b^2 - 60 = 0$$

$$b^2 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-60)}}{2(4)}$$

$$b^2 = \frac{1 \pm \sqrt{961}}{8}$$

$$= \frac{1 \pm 31}{8}$$

$$b^2 = \frac{1+31}{8}, \quad \frac{1-31}{8}$$



$$b^2 = \frac{32}{8}, \quad b^2 = \frac{-30}{8} \quad \text{(solution not possible)}$$

$$b^2 = 4$$

Required equation of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e; $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Vertices $(\pm a, 0) = (\pm 3, 0)$

Co-vertices $(0, \pm b) = (0, \pm 2)$

(vi) Vertices (0, ± 5), eccentricity = $\frac{3}{5}$

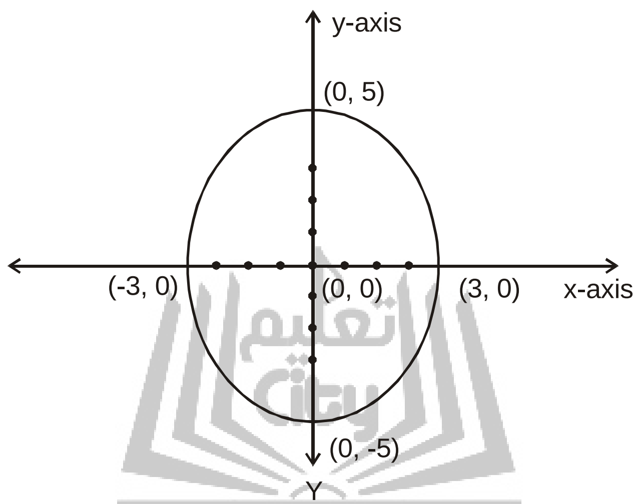
Solution:

Vertices (0, ± a) = (0, ± 5)

$a = 5 \qquad e = \frac{3}{5}$

$ae = 5 \times \frac{3}{5}$

=> $c = 3$



We know that

$c^2 = a^2 - b^2$

$(3)^2 = (5)^2 - b^2$

$9 = 25 - b^2$

$b^2 = 16$

Required equation of Ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

Co-vertices

(± b, 0) = (± 4, 0)

Foci = (0, ± c) = (0, ± 3)

(vii) Centre (0, 0) focus (0, – 3), vertex (0, 4)
(Lahore Board 2011)

Solution:

$(0, -c) = (0, -3) \qquad a = 4$

$c = 3$

We know that $e^2 = \dfrac{a^2 - b^2}{a^2}$

$(ae)^2 = a^2 - b^2$

$c^2 = a^2 - b^2$

$(3)^2 = 16 - b^2$

$9 = 16 - b^2$

$b^2 = 16 - 9 \quad \Rightarrow \quad b^2 = 7$

Required equation of ellipse is $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$

$\Rightarrow \quad \dfrac{y^2}{16} + \dfrac{x^2}{7} = 1$

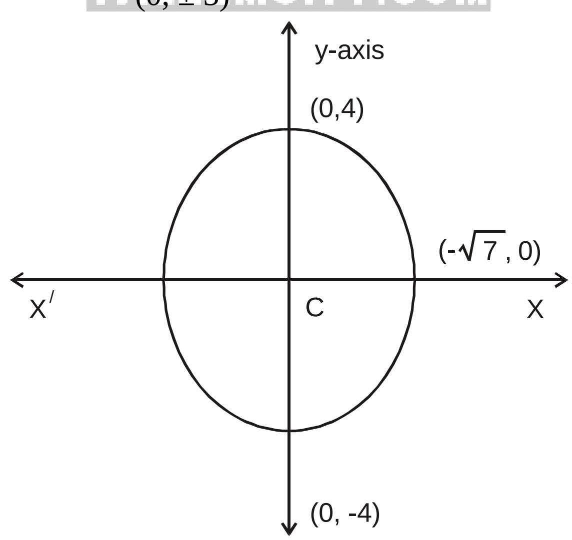
Its vertices are $(0, \pm a) = (0, \pm 4)$

Covertices $(\pm b, 0) = (\pm\sqrt{7}, 0)$

Coordinates of foci $= (0, \pm ae)$

$= (0, \pm 4\,\frac{3}{4})$

$= (0, \pm 3)$



(viii) Centre (2, 2) major axis parallel to y-axis and of length 8 units, minor axis parallel to x-axis and of length 6 units.

Solution:

Center (2, 2) also $2a = 8 \Rightarrow a = 4$

$2b = 6 \Rightarrow b = 3$

Equation of ellipse is $\dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$

$$\Rightarrow \frac{(y-2)^2}{16} + \frac{(x-2)^2}{9} = 1$$

Its vertices are $(0, \pm a) = (0, \pm 2)$

and $e^2 = \dfrac{a^2-b^2}{a^2} = \dfrac{16-9}{16} = \dfrac{7}{16} \Rightarrow e = \dfrac{\sqrt{7}}{4}$

Coordinates of foci $= (\pm ae, 0) = (\pm \dfrac{4\sqrt{7}}{4}, 0) = (\pm\sqrt{7}, 0)$

$(y-2,\ x-2) = (\pm\sqrt{7}, 0)$

$y = 2 \pm \sqrt{7}$, $x = 2$

Thus foci are $(2, 2+\sqrt{7})$ & $(2, 2-\sqrt{7})$

For vertices, we have $x - 2 = 0$ $y - 2 = \pm a$

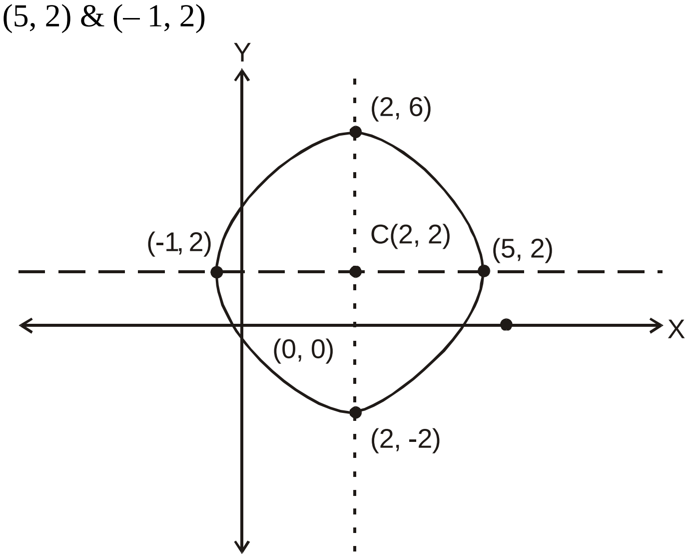
$x = 2$ $y = 2 \pm 4 \Rightarrow 6, -2$

So vertices are $(2, 6)$, $(2, -2)$

Next we have $x - 2 = \pm b$, $y - 2 = 0$

$x = 2 \pm 3$, $y = 2$

So covertices are $(5, 2)$ & $(-1, 2)$



(ix) Center (0, 0) symmetric with respect to both the axes and passing through the points (2, 3) and (6, 1).

Solution:

We know that equation of ellipse with center (0, 0) is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since it passes through the points (2, 3) & (6, 1)

$$\frac{4^2}{a^2} + \frac{9^2}{b^2} = 1 \qquad \text{(I)} \qquad\qquad \frac{36}{a^2} + \frac{1}{b^2} = 1 \qquad \text{(II)}$$

Subtracting

$$\begin{array}{rcl} 4b^2 + 9a^2 & = & a^2b^2 \\ -36b^2 \pm a^2 & = & -a^2b^2 \\ \hline -32b^2 + 8a^2 & = & 0 \end{array}$$

$$8a^2 = 32b^2$$

$$a^2 = 4b^2 \qquad \text{Put in (I)}$$

$$\frac{4}{4b^2} + \frac{9}{b^2} = 1$$

$$\frac{1+9}{b^2} = 1$$

$$\boxed{10 = b^2} \qquad \boxed{a^2 = 40}$$

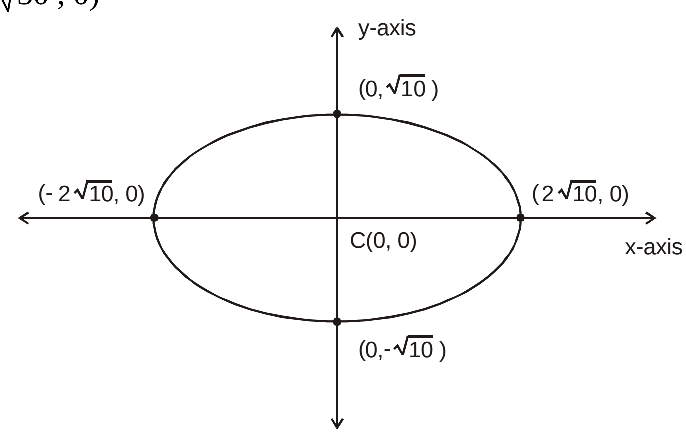
Required equation of ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1$$

Vertices are $(\pm a, 0) = (\pm\sqrt{40}, 0) = (\pm 2\sqrt{10}, 0)$

Covertices $(0, \pm b) = (0, \pm\sqrt{10})$

Foci $(\pm\sqrt{30}, 0)$



(x) Center (0, 0) major axis horizontal, the points (3, 1) (4, 0) lie on the graph.

Solution:

We know that equation of Ellipse with center (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since it passes though the points (3, 1) & (4, 0)

For (3, 1) For (4, 0)

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \qquad (1) \qquad \frac{16}{a^2} + \frac{0}{b^2} = 1 \qquad (2)$$

$\boxed{a^2 = 16}$ Put in (1)

$$\frac{9}{16} + \frac{1}{b^2} = 1$$

$$\frac{1}{b^2} = 1 - \frac{9}{16} = \frac{7}{16} \qquad \boxed{b^2 = \frac{16}{7}}$$

Required equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

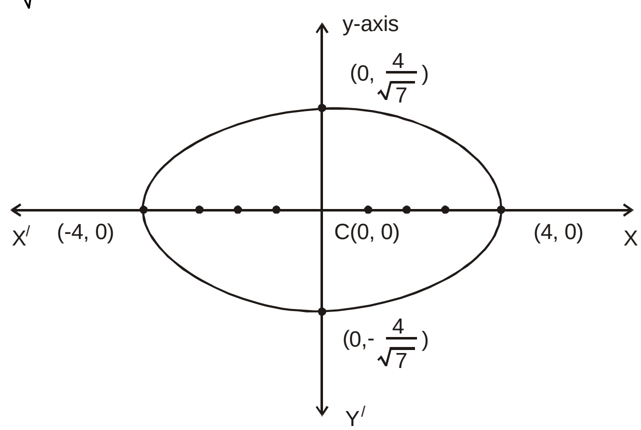
$$\frac{x^2}{16} + \frac{y^2}{16} = 1$$

$$\frac{x^2}{16} + \frac{7y^2}{16} = 1$$

Vertices $(\pm a, 0) = (\pm 4, 0)$

Covertices $(0, \pm b) = (0, \pm \frac{4}{\sqrt{7}})$

Foci $(\pm 4\sqrt{\frac{6}{7}}, 0)$



Q.2: Find the center, foci, eccentricity, vertices and directrix of the ellipse whose equation is given.

(i) $x^2 + 4y^2 = 16$ (Lahore Board 2009 (Supply))

Solution:

$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

Here $a^2 = 16$ $b^2 = 4$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

Foci are $= (\pm c, 0) = (\pm 2\sqrt{3}, 0)$

Vertices are $= (\pm a, 0) = (\pm 4, 0)$

Directrix are $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{4}{\frac{\sqrt{3}}{2}} = \pm \frac{8}{\sqrt{3}}$

Clearly center of ellipse is (0, 0)

(ii) $9x^2 + y^2 = 18$

Solution:

$$9x^2 + y^2 = 18$$

$$\frac{x^2}{2} + \frac{y^2}{18} = 1$$

$a^2 = 18$, $b^2 = 2$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{18-2}{18} = \frac{16}{18} = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$$

foci are $= (0, \pm c) = (0, \pm 4)$

vertices are $= (\pm a, 0) = (\pm 3\sqrt{2}, 0)$

directrix are $= y = \pm \frac{a}{e} = \pm \frac{3\sqrt{2}}{\frac{2\sqrt{2}}{3}} = \pm \frac{9}{2}$

Clearly center is (0, 0)

(iii) $\mathbf{25x^2 + 9y^2 = 225}$

Solution:

$$\frac{25x^2}{225} + \frac{9y^2}{225} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \text{(I)}$$

$a^2 = 25$ & $b^2 = 9$

Eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25 - 9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

foci $= (0, \pm c) = (0, \pm 4)$

vertices $= (0, \pm a) = (0, \pm 5)$

Center $= (0, 0)$

Directrix $y = \pm\dfrac{a}{e} = \pm\dfrac{5}{\frac{4}{5}} = \pm\dfrac{25}{4}$

(iv) $\mathbf{\dfrac{(2x-1)^2}{4} + \dfrac{(y+2)^2}{16} = 1}$

Solution:

Let $2x - 1 = X$, $y + 2 = Y$

Given equation becomes

$\dfrac{X^2}{4} + \dfrac{Y^2}{16} = 1$ Here $a^2 = 16$, $b^2 = 4$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{16 - 4}{16} = \frac{12}{16} = \frac{3}{4}$$

Eccentricity

$$e = \frac{\sqrt{3}}{2}$$

Center:- For center put $X = 0$, $Y = 0$

$2x - 1 = 0$, $y + 2 = 0$

$x = \dfrac{1}{2}$, $y = -2$

Required center is $\left(\dfrac{1}{2}, -2\right)$

Foci $(0, \pm ae) = \left(0, \pm 4\left(\dfrac{\sqrt{3}}{2}\right)\right)$

$(X, Y) = (0, \pm 2\sqrt{3})$

$(2x - 1, y + 2) = (0, \pm 2\sqrt{3})$

$2x - 1 = 0 \qquad y + 2 = \pm 2\sqrt{3}$

$x = \frac{1}{2} \qquad y = -2 \pm 2\sqrt{3}$

Required foci are $(\frac{1}{2}, -2 \pm 2\sqrt{3})$

Vertices $(0, \pm a) = (0, \pm 4)$

$(X, Y) = (0, \pm 4)$

$(2x - 1, y + 2) = (0, \pm 4)$

$2x - 1 = 0 \qquad y + 2 = \pm 4$

$x = \frac{1}{2} \qquad y = -2 \pm 4$

$y = -6, \quad 2$

Vertices are $\left(\frac{1}{2}, -6\right), \left(\frac{1}{2}, 2\right)$

Directrices $Y = \pm \frac{a}{e}$

$y + 2 = \pm \dfrac{4}{\frac{\sqrt{3}}{2}}$

$y + 2 = \pm \frac{8}{\sqrt{3}} \quad => \quad y = -2 \pm \frac{8}{\sqrt{3}}$

(v) $\mathbf{x^2 + 16x + 4y^2 - 16y + 76 = 0}$

Solution:

$x^2 + 16x + 4(y^2 - 4y) = -76$

$(x^2 + 16x + 64) + 4(y^2 - 4y + 4) = -76 + 64 + 16$

$(x + 8)^2 + 4(y - 2)^2 = 4$

$\frac{(x + 8)^2}{4} + \frac{4(y - 2)^2}{4} = \frac{4}{4}$

$\frac{(x + 8)^2}{4} + (y - 2)^2 = 1$

Let

$x + 8 = X, \qquad y - 2 = Y$

$$\frac{X^2}{4} + \frac{Y^2}{1} = 1 \quad \text{(an ellipse)}$$

$a^2 = 4$, $b^2 = 1$ $\quad e^2 = \frac{a^2 - b^2}{a^2} = \frac{4-1}{4} = \frac{3}{4}$

$\Rightarrow e = \frac{\sqrt{3}}{2}$

For the center Put $X = 0$, $Y = 0$

$x + 8 = 0$, $y - 2 = 0$

$x = -8$, $y = 2$

Required centre $(-8, 2)$

Foci $= (\pm ae, 0)$

$(X, Y) = \left(\pm 2\left(\frac{\sqrt{3}}{2}\right), 0\right)$

$(x + 8, y - 2) = (\pm\sqrt{3}, 0)$

$x + 8 = \pm\sqrt{3}$ $\quad y - 2 = 0$

$x = -8 \pm \sqrt{3}$ $\quad y = 2$

Required foci are $(-8 \pm \sqrt{3}, 2)$

Vertices are $= (\pm a, 0)$

$(X, Y) = (\pm a, 0)$

$(x + 8, y - 2) = (\pm 2, 0)$

$x + 8 = \pm 2$, $y - 2 = 0$

$x = -8 \pm 2$ $\quad y = 2$

Required vertices are $(-6, 2)$ & $(-10, 2)$

Directrix $X = \pm\frac{a}{e}$

$$x + 8 = \pm\frac{2}{\frac{\sqrt{3}}{2}} = \pm\frac{4}{\sqrt{3}}$$

$$x = -8 \pm \frac{4}{\sqrt{3}}$$

(vi) $\mathbf{25x^2 + 4y^2 - 250x - 16y + 541 = 0}$

Solution:

$25x^2 - 250x + 4y^2 + 16y = -541$

$25(x^2 - 10x) + 4(y^2 - 4y) = -541$

$25(x^2 - 10x + 25) + 4(y^2 - 4y + 4) = -541 + 625 + 16$

$25(x-5)^2 + 4(y-2)^2 = 100$

Dividing both sides by 100

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$$

Let

$x - 5 = X$, $y - 2 = Y$ above equation

Becomes

$$\frac{X^2}{4} + \frac{Y^2}{25} = 1 \quad \text{(an Ellipse)}$$

$a^2 = 25$, $b^2 = 4$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25-4}{25} = \frac{21}{25}$$

$$e = \frac{\sqrt{21}}{5}$$

For Center Put $X = 0$, $Y = 0$

$(x-5, y-2) = (0, 0)$

$x - 5 = 0$, $y - 2 = 0$

$x = 5$ $y = 2$

Center is (5, 2)

Foci $= (0, \pm ae) = (0, \pm 5\frac{\sqrt{21}}{5})$

$(X, Y) = (0, \pm\sqrt{21})$

$(x-5, y-2) = (0, \pm\sqrt{21})$

$x - 5 = 0$, $y - 2 = \pm\sqrt{21}$

$x = 5$, $y = 2 \pm\sqrt{21}$

Foci are $(5, 2 \pm\sqrt{21})$

Vertices $(0, \pm a) = (0, \pm 5)$

$(X, Y) = (0, \pm 5)$

$(x-5, y-2) = (0, \pm 5)$

$x - 5 = 0$, $y - 2 = \pm 5$

$x = 5$, $y = 2 \pm 5$

$y = 7, -3$

Vertices are (5, 7) & (5, – 3)

Directrix are

$$Y = \pm\frac{a}{e}$$

$$y - 2 = \pm\frac{5}{\frac{\sqrt{21}}{5}}$$

$$y = 2 \pm \frac{25}{\sqrt{21}}$$

Q.3: Let a be a + ve number and 0 < c < a. Let F(– c, 0) & F′(c > 0) be two given points. Prove that the locus of points p(x, y) such that |PF| + |PF′| = 2a is an ellipse.

Solution:

Given P(x, y) , F(– c, 0) , F′(c , 0)

$\therefore \quad |PF| + |PF| = 2a$

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Squaring on both sides

$$[\sqrt{(x+c)^2 + y^2}\,]^2 = [2a - \sqrt{(x-c)^2 + y^2}\,]^2$$

$$(x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x+c)^2 + y^2}$$

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

(Dividing by 4)

$$(cx - a^2) = -a\sqrt{(x-c)^2 + y^2}$$

Again Squaring

$$(cx - a^2)^2 = [-a\sqrt{(x-c)^2 + y^2}\,]^2$$

$$c^2x^2 + a^4 - 2cxa^2 = a^2(x^2 + c^2 - 2cx + y^2)$$

$$c^2x^2 + a^4 - 2cxa^2 = a^2x^2 + a^2c^2 - 2cxa^2 + a^2y^2$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

Dividing throughout by $c^2 - a^2$

$$x^2 - \frac{a^2y^2}{c^2 - a^2} = a^2 \qquad (1)$$

We know that for ellipse $c^2 = a^2 - b^2 \quad \Rightarrow \quad c^2 - a^2 = -b^2$ put in (1)

$$x^2 - \frac{a^2y^2}{-b^2} = a^2$$

$$x^2 + \frac{a^2y^2}{b^2} = a^2$$

(Dividing throughout by a^2)

$$\frac{x^2}{a^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2}{a^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{which is an ellipse}$$

Q.4: Use problem 3 to find equation of the ellipse as locus of points P(x, y) such that the sum of the distances from P to the points (0, 0) & (1, 1) is 2.

Solution:

Given P(x, y), F(0, 0) & F′(1, 1) Also given that 2a = 2

For ellipse we know that

$$|PF| + |PF'| = 2a$$

$$\sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2$$

$$\sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2$$

$$\sqrt{(x-1)^2 + (y-1)^2} = 2 - \sqrt{x^2 + y^2} \quad \text{Squaring}$$

$$(x-1)^2 + (y-1)^2 = 4 + (x^2 + y^2) - 4\sqrt{x^2 + y^2}$$

$$x^2 + 1 - 2x + y^2 + 1 - 2y = 4 + x^2 + y^2 - 4\sqrt{x^2 + y^2}$$

$$-2x - 2y - 2 = -4\sqrt{x^2 + y^2}$$

$$x + y + 1 = 2\sqrt{x^2 + y^2} \quad \text{(Dividing both sides by 2)}$$

Again squaring

$$(x + y + 1)^2 = (2\sqrt{x^2 + y^2})^2$$

$$x^2 + y^2 + 1 + 2xy + 2y + 2x = 4(x^2 + y^2)$$

$$4x^2 + 4y^2 - x^2 - y^2 - 1 - 2xy - 2y - 2x = 0$$

$$3x^2 + 3y^2 - 2xy - 2x - 2y - 1 = 0 \qquad \text{required ellipse}$$

Q.5: Prove that latusrectim of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$

Solution:

The given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

From the figure, the points A(c, h) & B(c, – h) lies on ellipse (1) therefore.

For A(c, h) equation (1) becomes

$$\frac{c^2}{a^2} + \frac{h^2}{b^2} = 1$$

$$\frac{h^2}{b^2} = 1 - \frac{c^2}{a^2}$$

$$\frac{h^2}{b^2} = \frac{a^2 - c^2}{a^2} \qquad (2)$$

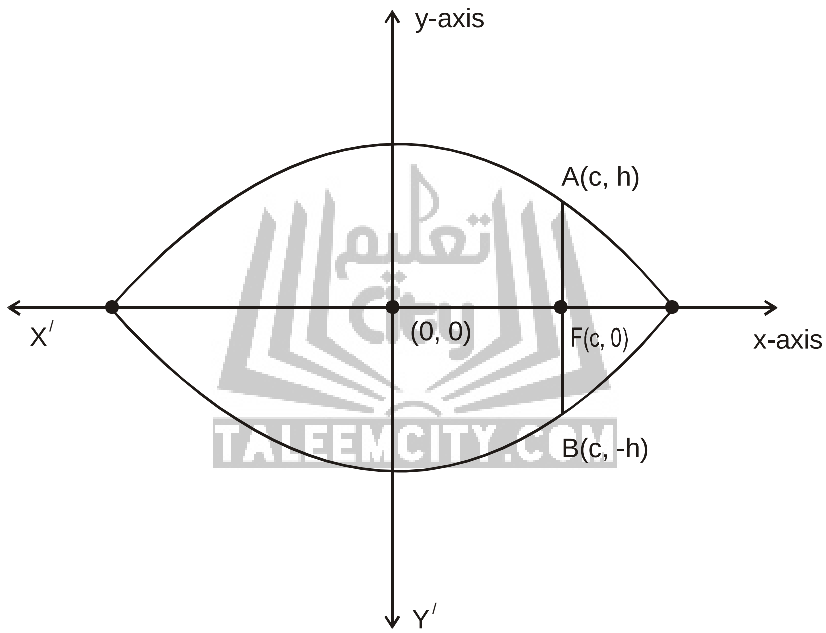
For ellipse, we know that

$$c^2 = a^2 - b^2$$

=> $b^2 = a^2 - c^2$ Put in (2)

$$\frac{h^2}{b^2} = \frac{b^2}{a^2}$$

$$h^2 = \frac{b^4}{a^2} \quad \Rightarrow \quad h = \pm\frac{b^2}{a}$$



Points A & B becomes

$$A\left(c, \frac{b^2}{a}\right) \text{ and } B\left(c, -\frac{b^2}{a}\right)$$

Length of Latus rectum AB $= \sqrt{(c-c)^2 + \left(\frac{-b^2}{a} - \frac{b^2}{a}\right)^2}$

$$= \sqrt{\left(\frac{-2b^2}{a}\right)^2} = \sqrt{\frac{4b^4}{a^2}}$$

$$= \frac{2b^2}{a} \quad \text{Hence proved.}$$

Q.6: The major axis of an ellipse in standard form lies along the x-axis and has length $4\sqrt{2}$. The distance between the foci equals the length of minor axis. Write an equation of the ellipse.

Solution:

Since the major axis of the ellipse lies along x–axis so its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{.......} \quad (1)$$

By the given condition $2a = 4\sqrt{2}$

$\Rightarrow \quad a = 2\sqrt{2}$

We know that

Distance between foci = length of minor axis

$2c = 2b$

$\Rightarrow \quad c = b$

Since for ellipse $c^2 = a^2 - b^2$

$b^2 = a^2 - b^2$

$2b^2 = a^2 \quad \Rightarrow \quad b^2 = \frac{a^2}{2}$

$\Rightarrow \quad b^2 = \frac{(2\sqrt{2})^2}{2} = \frac{8}{2} = 4$

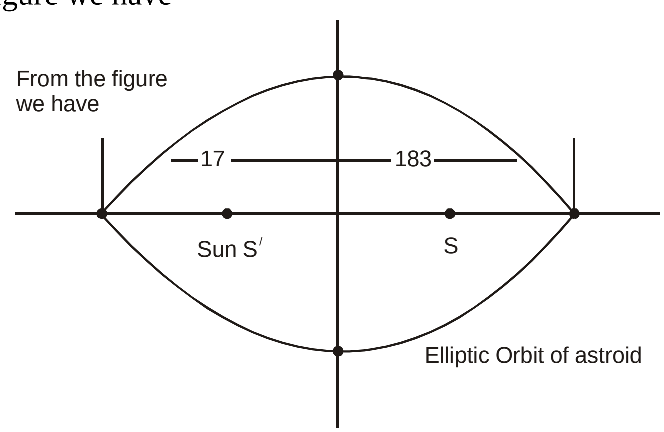
Putting values of a & b in (1)

$$\frac{x^2}{8} + \frac{y^2}{4} = 1 \qquad \text{Required equation of ellipse}$$

Q.7: An astroid has elliptic orbit with the sun at one focus. Its distance from the sun ranges from 17 million miles to 183 million miles. Write on equation of the orbit of the astroid.

Solution: (Lahore Board 2004)

From the figure we have



Greatest distance of astroid from the sun is a + c = 183 (1)

And least distance of astroid from the sun is

$a - c = 17$ (2)

Adding (1) & (2)

$$\begin{aligned} a + c &= 183 \\ a - c &= 17 \\ \hline 2a &= 200 \\ a &= 100 \quad \text{Put in (1)} \\ 100 + c &= 183 \\ c &= 83 \end{aligned}$$

For ellipse, we know that

$c^2 = a^2 - b^2 \quad \Rightarrow \quad b^2 = a^2 - c^2$

$\Rightarrow \quad b^2 = (100)^2 - (83)^2$

$b^2 = 3111$

Required equation of the orbit of astroid is

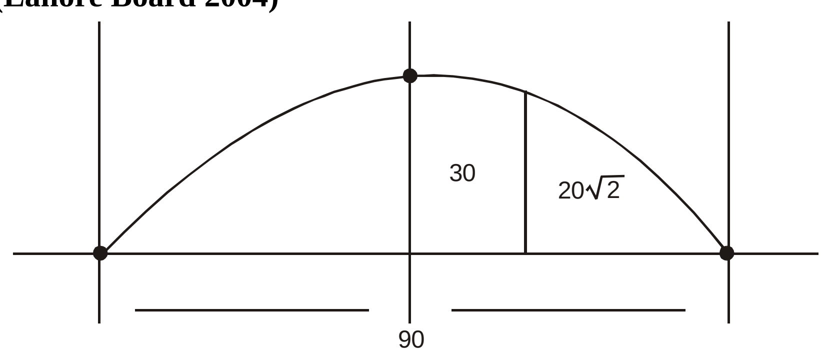
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(100)^2} + \frac{y^2}{3111} = 1$$

$$\frac{x^2}{10000} + \frac{y^2}{3111} = 1 \quad \text{Ans.}$$

Q.8: An arch in the shape of semi ellipse is 90m wide at the base and 30 m high at the center. At what distance from the center is the arch $20\sqrt{2}$ m high?

Solution: (Lahore Board 2004)



Let the equation of the semi-elliptic are be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .... (1)

Since the arch is 90m wide at the base

so $2a = 90 \Rightarrow a = 45$

Since arch is 30 m high at the center, so

$b = 30$, putting a & b in (1)

$$\frac{x^2}{(45)^2} + \frac{y^2}{(30)^2} = 1 \quad ....(2)$$

Let d be the required distance from the center when arch is $20\sqrt{2}$ m high.

Putting $x = d$, $y = 20\sqrt{2}$ in (2)

$$\frac{d^2}{(45)^2} + \frac{(20\sqrt{2})^2}{(30)^2} = 1$$

$$\frac{d^2}{2025} + \frac{800}{900} = 1$$

$$\frac{d^2}{2025} = 1 - \frac{800}{900}$$

$$d^2 = \frac{100}{900} \times 2025$$

$$d^2 = \frac{(45)^2}{(3)^2}$$

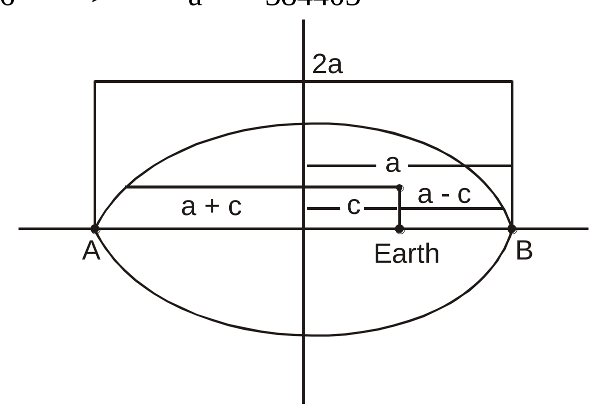
$$d = \frac{45}{3} = 15 \text{ meter}$$

Q.9: The moon orbits the earth in an elliptic path with earth at one focus. The major and minor axes of the orbit are 768, 806 km & 767, 746 km respectively. Find the greatest and least distances in astronomy called apogee & perigee) of the moon from the earth.

Solution:

By the given conditions

$2a = 768806 \Rightarrow a = 384403$



$2b = 767746 \Rightarrow b = 383874$

Since $\sqrt{c^2} = \sqrt{a^2 - b^2}$

$c = \sqrt{(384403)^2 - (383874)^2}$

$c = 20179$ km

The greatest distance

$a + c = 384403 + 20179$

$= 404582$ km

The least distance

$a - c = 384403 - 20179$

$= 364224$ km

Hyperbola

Let $e > 1$ and F be a fixed point and L be line not containing F. Also let P(x, y) be any point in the plane and |PM| be the perpendicular distance of P from L. The set of all the points P(x, y) such that $\frac{|PF|}{|PM|} = e > 1$ is called hyperbola.

Key points about Hyperbola

(1) Equation (Standard form)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(2) Eccentricity $e^2 = \frac{a^2+b^2}{a^2}$

$(ae)^2 = a^2 + b^2 \Rightarrow c^2 = a^2 + b^2$

(3) Foci $(\pm ae, 0)$

(4) Directrix $x = \pm \frac{a}{e}$

(5) Center (0, 0)

(6) Vertices $(\pm a, 0)$

(7) Covertices $(0, \pm b)$

(1) (Lahore Board 2009)

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(2) Eccentricity $e^2 = \frac{a^2+b^2}{a^2}$

$(ae)^2 = a^2 + b^2$

$c^2 = a^2 + b^2$

(3) Foci $(0, \pm ae)$

(4) Directrix $y = \pm \frac{a}{e}$

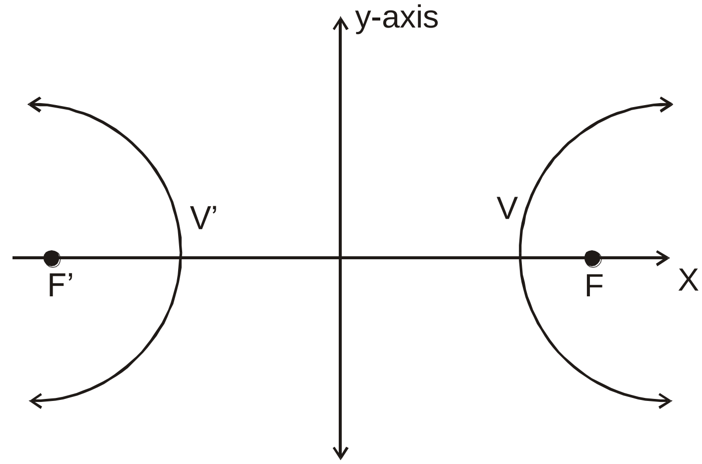
(5) Center (0, 0)

(6) Vertices $(0, \pm a)$

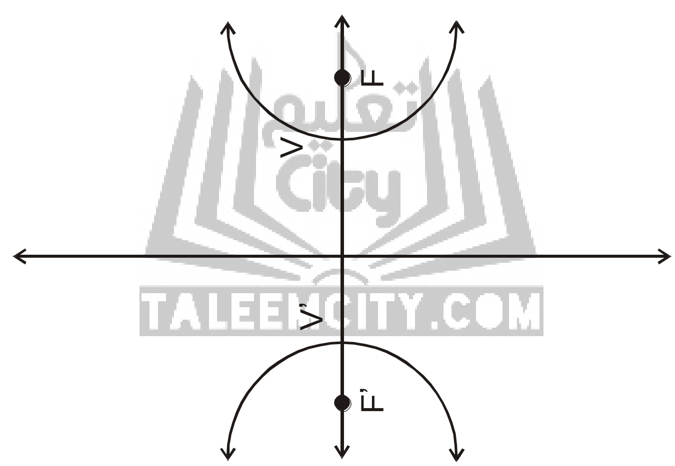
(7) Covertices $(\pm b, 0)$

Length of latus rectum is $\frac{2b^2}{a}$.

If center is not (0, 0) then equations become



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \& \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



EXERCISE 6.6

Q.1: Find an equation of the hyperbola with given data. Sketch the graph of each.

(i) Center (0, 0) Focus (6, 0) , Vertex (4, 0)

Solution:

Given $(ae, 0) = (6, 0)$, $(a, 0) = (4, 0)$

$(c, 0) = (6, 0)$ $a = 4$

$c = 6$