

Question # 1

Prove that

(i) $\sin(180^\circ + \theta) = -\sin \theta$

(ii) $\cos(180^\circ + \theta) = -\cos \theta$

(iii) $\tan(270^\circ - \theta) = \cot \theta$

(iv) $\cos(\theta - 180^\circ) = -\cos \theta$

(v) $\cos(270^\circ + \theta) = \sin \theta$

(vi) $\sin(\theta + 270^\circ) = -\cos \theta$

(vii) $\tan(180^\circ + \theta) = \tan \theta$

(viii) $\cos(360^\circ - \theta) = \cos \theta$

Solution

$$\begin{aligned}
 \text{(i) L.H.S} &= \sin(180 + \theta) = \sin 180 \cos \theta + \cos 180 \sin \theta \\
 &= \sin(0) \cos \theta + (-1) \sin \theta = 0 - \sin \theta = -\sin \theta = \text{R.H.S}
 \end{aligned}$$

(ii) *Do youself*

$$\begin{aligned}
 \text{(iii) L.H.S} &= \tan(270^\circ - \theta) = \frac{\tan 270^\circ - \tan \theta}{1 + \tan 270^\circ \tan \theta} \\
 &= \frac{\tan 270^\circ \left(1 - \frac{\tan \theta}{\tan 270^\circ}\right)}{\tan 270^\circ \left(\frac{1}{\tan 270^\circ} + \tan \theta\right)} = \frac{\left(1 - \frac{\tan \theta}{\infty}\right)}{\left(\frac{1}{\infty} + \tan \theta\right)} \\
 &= \frac{(1-0)}{(0+\tan \theta)} = \frac{1}{\tan \theta} = \cot \theta = \text{R.H.S}
 \end{aligned}$$

*Remaining do yourself.***Question # 2**

Find the values of the following:

(i) $\sin 15^\circ$

(ii) $\cos 15^\circ$

(iii) $\tan 15^\circ$

Solution

(i) Since $15 = 45 - 30$

So $\sin 15^\circ = \sin(45 - 30) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) Since $15 = 45 - 30$

So $\cos 15^\circ = \cos(45 - 30) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) Since $15 = 45 - 30$

So $\tan 15^\circ = \tan(45 - 30) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + (1) \left(\frac{1}{\sqrt{3}} \right)} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\cancel{\sqrt{3}-1}}{\cancel{\sqrt{3}+1}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

For (iv), (v) and (vi), we have hint:

Hint: Use $105 = 60 + 45$ in these questions

Question # 3

Prove that:

$$(i) \sin(45^\circ + \alpha) = \frac{1}{\sqrt{2}}(\sin \alpha + \cos \alpha) \quad (ii) \cos(45^\circ + \alpha) = \frac{1}{\sqrt{2}}(\cos \alpha - \sin \alpha)$$

Solution

$$(i) \text{ L.H.S} = \sin(45 + \alpha)$$

$$\begin{aligned} &= \sin 45^\circ \cos \alpha + \cos 45^\circ \sin \alpha = \left(\frac{1}{\sqrt{2}} \cos \alpha + \frac{1}{\sqrt{2}} \sin \alpha \right) \\ &= \frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha) = \frac{1}{\sqrt{2}} (\sin \alpha + \cos \alpha) = \text{R.H.S} \end{aligned}$$

$$(ii) \text{ Do yourself as above}$$

Question # 4

Prove that:

$$(i) \tan(45 + A) \tan(45 - A) = 1$$

$$(ii) \tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$$

$$(iii) \sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos \theta$$

$$(iv) \frac{\sin \theta - \cos \theta \tan \frac{\theta}{2}}{\cos \theta + \sin \theta \tan \frac{\theta}{2}} = \tan \frac{\theta}{2}$$

$$(v) \frac{1 - \tan \theta \tan \varphi}{1 + \tan \theta \tan \varphi} = \frac{\cos(\theta + \varphi)}{\cos(\theta - \varphi)}$$

Solution

$$(i) \text{ L.H.S} = \tan(45 + A) \tan(45 - A)$$

$$\begin{aligned} &= \left(\frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} \right) \left(\frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A} \right) \\ &= \left(\frac{1 + \tan A}{1 - (1) \tan A} \right) \left(\frac{1 - \tan A}{1 + (1) \tan A} \right) = \left(\frac{1 + \tan A}{1 - \tan A} \right) \left(\frac{1 - \tan A}{1 + \tan A} \right) = 1 = \text{R.H.S} \end{aligned}$$

$$(ii) \text{ L.H.S} = \tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right)$$

$$\begin{aligned} &= \left(\frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right) + \left(\frac{\tan \frac{3\pi}{4} + \tan \theta}{1 - \tan \frac{3\pi}{4} \tan \theta} \right) \\ &= \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) + \left(\frac{-1 + \tan \theta}{1 - \tan \theta} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1-\tan\theta}{1+(1)\tan\theta} \right) + \left(\frac{-1+\tan\theta}{1-(-1)\tan\theta} \right) \\
&= \left(\frac{1-\tan\theta}{1+\tan\theta} \right) + \left(\frac{-1+\tan\theta}{1+\tan\theta} \right) \\
&= \frac{1-\tan\theta-1+\tan\theta}{1+\tan\theta} = \frac{0}{1+\tan\theta} = 0 = \text{R.H.S}
\end{aligned}$$

(iii) L.H.S = $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right)$

$$\begin{aligned}
&= \left(\sin\theta \cos\frac{\pi}{6} + \cos\theta \sin\frac{\pi}{6} \right) + \left(\cos\theta \cos\frac{\pi}{3} - \sin\theta \sin\frac{\pi}{3} \right) \\
&= \left(\sin\theta \frac{\sqrt{3}}{2} + \cos\theta \frac{1}{2} \right) + \left(\cos\theta \frac{1}{2} - \sin\theta \frac{\sqrt{3}}{2} \right) \\
&= \frac{\sqrt{3}}{2} \sin\theta + \frac{1}{2} \cos\theta + \frac{1}{2} \cos\theta - \frac{\sqrt{3}}{2} \sin\theta = \cos\theta = \text{R.H.S.}
\end{aligned}$$

(iv) L.H.S = $\frac{\sin\theta - \cos\theta \tan\frac{\theta}{2}}{\cos\theta + \sin\theta \tan\frac{\theta}{2}}$

$$\begin{aligned}
&= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} - \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \frac{\sin\theta \cos\frac{\theta}{2} - \cos\theta \sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\
&= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\cos\theta \cos\frac{\theta}{2} + \sin\theta \sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\
&= \frac{\sin\theta \cos\frac{\theta}{2} - \cos\theta \sin\frac{\theta}{2}}{\cos\theta \cos\frac{\theta}{2} + \sin\theta \sin\frac{\theta}{2}} = \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\cos\left(\theta - \frac{\theta}{2}\right)} = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \tan\frac{\theta}{2} = \text{R.H.S}
\end{aligned}$$

(v) L.H.S = $\frac{1-\tan\theta\tan\varphi}{1+\tan\theta\tan\varphi}$

$$\begin{aligned}
&= \frac{1 - \frac{\sin\theta}{\cos\theta} \frac{\sin\varphi}{\cos\varphi}}{1 + \frac{\sin\theta}{\cos\theta} \frac{\sin\varphi}{\cos\varphi}} = \frac{\cos\theta \cos\varphi - \sin\theta \sin\varphi}{\cos\theta \cos\varphi + \sin\theta \sin\varphi} \\
&= \frac{\cos\theta \cos\varphi - \sin\theta \sin\varphi}{\cos\theta \cos\varphi + \sin\theta \sin\varphi} = \frac{\cos(\theta + \varphi)}{\cos(\theta - \varphi)} = \text{R.H.S}
\end{aligned}$$

Question # 5

$$\text{Show that: } \cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta = \cos^2 \beta - \sin^2 \alpha$$

Solution

Question # 6

Do yourself as above

Hint: Just open the formulas

Question # 7

Show that

$$\begin{aligned}
 \text{(i)} \quad & \cot(\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta} \\
 \text{(ii)} \quad & \cot(\alpha - \beta) = \frac{\cot \alpha \cot \beta + 1}{\cot \beta - \cot \alpha} \\
 \text{(iii)} \quad & \frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}
 \end{aligned}$$

Solution

$$\begin{aligned}
 \text{(i) L.H.S} &= \cot(\alpha + \beta) = \frac{1}{\tan(\alpha + \beta)} = \frac{1}{\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}} \\
 &= \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} = \frac{\tan \alpha \tan \beta \left(\frac{1}{\tan \alpha \tan \beta} - 1 \right)}{\tan \alpha \tan \beta \left(\frac{1}{\tan \beta} + \frac{1}{\tan \alpha} \right)} \\
 &= \frac{\cot \alpha \cot \beta - 1}{\cot \beta + \cot \alpha} = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta} = \text{R.H.S}
 \end{aligned}$$

(ii)

Do yourself as above

(iii)

$$\text{L.H.S} = \frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{\frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta}} = \frac{\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}$$

$$= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \cos \beta - \cos \alpha \sin \beta} = \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \text{R.H.S}$$

Question # 8

If $\sin \alpha = \frac{4}{5}$ and $\cos \alpha = \frac{40}{41}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.

Show that $\sin(\alpha - \beta) = \frac{133}{205}$

Solution Since $\sin \alpha = \frac{4}{5}$; $0 < \alpha < \frac{\pi}{2}$
 $\cos \alpha = \frac{40}{41}$; $0 < \beta < \frac{\pi}{2}$

Now

$$\begin{aligned}\cos^2 \alpha &= 1 - \sin^2 \alpha \\ \Rightarrow \cos \alpha &= \pm \sqrt{1 - \sin^2 \alpha}\end{aligned}$$

Since terminal ray of α is in the first quadrant so value of cos is +ive

$$\begin{aligned}\cos \alpha &= \sqrt{1 - \sin^2 \alpha} \\ \Rightarrow \cos \alpha &= \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} \Rightarrow \boxed{\cos \alpha = \frac{3}{5}}\end{aligned}$$

Also

$$\sin^2 \beta = 1 - \cos^2 \beta \Rightarrow \sin \beta = \pm \sqrt{1 - \cos^2 \beta}$$

Since terminal ray of β is in the first quadrant so value of sin is +ive

$$\begin{aligned}\sin \beta &= \sqrt{1 - \cos^2 \beta} \\ \Rightarrow \sin \beta &= \sqrt{1 - \left(\frac{40}{41}\right)^2} = \sqrt{1 - \frac{1600}{1681}} = \sqrt{\frac{81}{1681}} \Rightarrow \boxed{\sin \beta = \frac{9}{41}}\end{aligned}$$

Now

$$\begin{aligned}\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= \left(\frac{4}{5}\right)\left(\frac{40}{41}\right) - \left(\frac{3}{5}\right)\left(\frac{9}{41}\right) = \frac{160}{205} - \frac{27}{205} = \frac{133}{205}\end{aligned}$$

i.e. $\sin(\alpha - \beta) = \frac{133}{205}$ *Proved*

Question # 9

If $\sin \alpha = \frac{4}{5}$ and $\sin \beta = \frac{12}{13}$ where $\frac{\pi}{2} < \alpha < \pi$ and $\frac{\pi}{2} < \beta < \pi$. Find

- | | | | | | |
|------|------------------------|------|------------------------|-------|------------------------|
| (i) | $\sin(\alpha + \beta)$ | (ii) | $\cos(\alpha + \beta)$ | (iii) | $\tan(\alpha + \beta)$ |
| (iv) | $\sin(\alpha - \beta)$ | (v) | $\cos(\alpha - \beta)$ | (vi) | $\tan(\alpha - \beta)$ |

In which quadrant do the terminal sides of the angles of measures $(\alpha + \beta)$ and $(\alpha - \beta)$ lie?

Solution

$$\text{Since } \sin \alpha = \frac{4}{5} ; \quad \frac{\pi}{2} < \alpha < \pi$$

$$\sin \beta = \frac{12}{13} ; \quad \frac{\pi}{2} < \beta < \pi$$

$$\text{Since } \cos^2 \alpha = 1 - \sin^2 \alpha \Rightarrow \cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}$$

As terminal ray of α lies in the IIInd quadrant so value of cos is -ive

$$\begin{aligned} \cos \alpha &= -\sqrt{1 - \sin^2 \alpha} \\ \Rightarrow \cos \alpha &= -\sqrt{1 - \left(\frac{4}{5}\right)^2} = -\sqrt{1 - \frac{16}{25}} = -\sqrt{\frac{9}{25}} \Rightarrow \boxed{\cos \alpha = -\frac{3}{5}} \end{aligned}$$

Now

$$\begin{aligned} \cos^2 \beta &= 1 - \sin^2 \beta \\ \Rightarrow \cos \beta &= \pm \sqrt{1 - \sin^2 \beta} \end{aligned}$$

As terminal ray of β lies in the IIInd quadrant so value of cos is -ive

$$\begin{aligned} \cos \beta &= -\sqrt{1 - \sin^2 \beta} \\ \Rightarrow \cos \beta &= -\sqrt{1 - \left(\frac{12}{13}\right)^2} = -\sqrt{1 - \frac{144}{169}} = -\sqrt{\frac{25}{169}} \Rightarrow \boxed{\cos \beta = -\frac{5}{13}} \end{aligned}$$

$$\begin{aligned} \text{(i)} \quad \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) = -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65} \end{aligned}$$

$$\text{(iii)} \quad \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{-\frac{56}{65}}{-\frac{33}{65}} = \frac{56}{33}$$

(iv), (v) & (vi) *Do yourself as above*

Since $\sin(\alpha + \beta)$ is -ive so terminal are of $\alpha + \beta$ is in IIIrd or IVth quadrant and $\cos(\alpha + \beta)$ is -ive so terminal are of $\alpha + \beta$ is in IIInd or IIIrd quadrant therefore terminal ray of $\alpha + \beta$ lies in the IIIrd quadrant.

Similarly after solving (iv), (v) & (vi) find quadrant for $\alpha - \beta$ yourself.

Question # 10

Find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, given that

(i) $\tan \alpha = \frac{3}{4}$, $\sin \beta = \frac{5}{13}$ and neither the terminal side of the angle of measure α nor that of β is in the I quadrant.

(ii) $\tan \alpha = -\frac{15}{8}$, $\sin \beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the IV quadrant.

Solution

$$(i) \quad \text{Since } \tan \alpha = \frac{3}{4}$$

As $\tan \alpha$ is +ive and terminal arm of α is not in the Ist quad. Therefor it lies in IIIrd quad.

Now

$$\begin{aligned} \sec^2 \alpha &= 1 + \tan^2 \alpha \\ \Rightarrow \sec \alpha &= \pm \sqrt{1 + \tan^2 \alpha} \end{aligned}$$

Since terminal arm of α is in the IIIrd quad., therefor value of sec is -ive

$$\begin{aligned} \sec \alpha &= -\sqrt{1 + \tan^2 \alpha} \\ \Rightarrow \sec \alpha &= -\sqrt{1 + \left(\frac{3}{4}\right)^2} = -\sqrt{1 + \frac{9}{16}} = -\sqrt{\frac{25}{16}} \Rightarrow \sec \alpha = -\frac{5}{4} \end{aligned}$$

$$\text{Now } \cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{-\frac{5}{4}} \Rightarrow \boxed{\cos \alpha = -\frac{4}{5}}$$

$$\begin{aligned} \text{Now } \frac{\sin \alpha}{\cos \alpha} &= \tan \alpha \Rightarrow \sin \alpha = \tan \alpha \cos \alpha \\ \Rightarrow \sin \alpha &= \left(\frac{3}{4}\right)\left(-\frac{4}{5}\right) \Rightarrow \boxed{\sin \alpha = -\frac{3}{5}} \end{aligned}$$

$$\text{Since } \cos \beta = \frac{5}{13}$$

As $\cos \beta$ is +ive and terminal arm of β is not in the Ist quad., therefore it lies in IVth quad.

$$\begin{aligned} \text{Now } \sin^2 \beta &= 1 - \cos^2 \beta \\ \Rightarrow \sin \beta &= \pm \sqrt{1 - \cos^2 \beta} \end{aligned}$$

Since terminal ray of β is in the fourth quadrant so value of sin is -ive

$$\begin{aligned} \sin \beta &= -\sqrt{1 - \cos^2 \beta} \\ \Rightarrow \sin \beta &= -\sqrt{1 - \left(\frac{5}{13}\right)^2} = -\sqrt{1 - \frac{25}{169}} = -\sqrt{\frac{144}{169}} \Rightarrow \boxed{\sin \beta = -\frac{12}{13}} \end{aligned}$$

$$\text{Now } \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) = -\frac{3}{13} + \frac{48}{65} \Rightarrow \boxed{\sin(\alpha + \beta) = \frac{33}{65}}$$

And $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$$= \left(-\frac{4}{5} \right) \left(\frac{5}{13} \right) - \left(-\frac{3}{5} \right) \left(-\frac{12}{13} \right) = -\frac{4}{13} - \frac{36}{65} \rightarrow \boxed{\cos(\alpha + \beta) = -\frac{56}{65}}$$

(ii)

Do yourself as above

Question # 11

Prove that: $\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan 37^\circ$

Solution

$$\begin{aligned} \text{R.H.S} &= \tan 37^\circ = \tan(45^\circ - 8^\circ) && \because 37 = 45 - 8 \\ &= \frac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ} = \frac{1 - \tan 8^\circ}{1 + (1)\tan 8^\circ} \\ &= \frac{1 - \frac{\sin 8^\circ}{\cos 8^\circ}}{1 + \frac{\sin 8^\circ}{\cos 8^\circ}} = \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \text{L.H.S} \end{aligned}$$

Question # 12

If α, β, γ are the angles of a triangle ABC , show that

$$\cot \frac{\beta}{2} + \cot \frac{\alpha}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

Solution

Since α, β and γ are angles of triangle therefore

$$\alpha + \beta + \gamma = 180^\circ \Rightarrow \alpha + \beta = 180^\circ - \gamma$$

$$\Rightarrow \frac{\alpha + \beta}{2} = \frac{180^\circ - \gamma}{2} \Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ - \frac{\gamma}{2}$$

$$\text{Now } \tan \left(\frac{\alpha}{2} + \frac{\beta}{2} \right) = \tan \left(90^\circ - \frac{\gamma}{2} \right)$$

$$\Rightarrow \frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} = \cot \frac{\gamma}{2} \quad \therefore \tan \left(90^\circ - \frac{\gamma}{2} \right) = \cot \frac{\gamma}{2}$$

$$\Rightarrow \frac{\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \left(\frac{1}{\tan \frac{\beta}{2}} + \frac{1}{\tan \frac{\alpha}{2}} \right)}{\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \left(\frac{1}{\tan \frac{\alpha}{2} \tan \frac{\beta}{2}} - 1 \right)} = \cot \frac{\gamma}{2} \quad \Rightarrow \frac{\cot \frac{\beta}{2} + \cot \frac{\alpha}{2}}{\cot \frac{\alpha}{2} \cot \frac{\beta}{2} - 1} = \cot \frac{\gamma}{2}$$

$$\Rightarrow \cot \frac{\beta}{2} + \cot \frac{\alpha}{2} = \cot \frac{\gamma}{2} \left(\cot \frac{\alpha}{2} \cot \frac{\beta}{2} - 1 \right)$$

$$\begin{aligned}\Rightarrow \cot\frac{\beta}{2} + \cot\frac{\alpha}{2} &= \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2} - \cot\frac{\gamma}{2} \\ \Rightarrow \cot\frac{\beta}{2} + \cot\frac{\alpha}{2} + \cot\frac{\gamma}{2} &= \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}\end{aligned}$$

Question # 13

If $\alpha + \beta + \gamma = 180^\circ$, show that
 $\cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot \alpha = 1$

Solution

Since α, β and γ are angles of triangle therefore

$$\alpha + \beta + \gamma = 180 \Rightarrow \alpha + \beta = 180 - \gamma$$

Now $\tan(\alpha + \beta) = \tan(180 - \gamma)$

$$\begin{aligned}\Rightarrow \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} &= \tan(2(90) - \gamma) \\ \Rightarrow \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} &= -\tan \gamma \\ \Rightarrow \tan \alpha + \tan \beta &= -\tan \gamma(1 - \tan \alpha \tan \beta) \\ \Rightarrow \tan \alpha + \tan \beta &= -\tan \gamma + \tan \alpha \tan \beta \tan \gamma \\ \Rightarrow \tan \alpha + \tan \beta + \tan \gamma &= \tan \alpha \tan \beta \tan \gamma\end{aligned}$$

Dividing through out by $\tan \alpha \tan \beta \tan \gamma$

$$\begin{aligned}\frac{\tan \alpha}{\tan \alpha \tan \beta \tan \gamma} + \frac{\tan \beta}{\tan \alpha \tan \beta \tan \gamma} + \frac{\tan \gamma}{\tan \alpha \tan \beta \tan \gamma} &= \frac{\tan \alpha \tan \beta \tan \gamma}{\tan \alpha \tan \beta \tan \gamma} \\ \Rightarrow \cot \beta \cot \gamma + \cot \alpha \cot \gamma + \cot \alpha \cot \beta &= 1 \\ \Rightarrow \cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot \alpha &= 1\end{aligned}$$

Question # 14

Express the following in the form $r \sin(\theta + \phi)$ or $r \sin(\theta - \phi)$, where terminal sides of the angles of measure θ and ϕ are in the first quadrant:

- (i) $12 \sin \theta + 5 \cos \theta$ (ii) $3 \sin \theta - 4 \cos \theta$ (iii) $\sin \theta - \cos \theta$
 (iv) $5 \sin \theta - 4 \cos \theta$ (v) $\sin \theta + \cos \theta$

Solution

(i) $12 \sin \theta + 5 \cos \theta$

Let $12 = r \cos \phi$ and $5 = r \sin \phi$

Squaring and adding

$$\begin{aligned}(12)^2 + (5)^2 &= r^2 \cos^2 \phi + r^2 \sin^2 \phi \\ \Rightarrow 144 + 25 &= r^2 (\cos^2 \phi + \sin^2 \phi) \\ \Rightarrow 169 &= r^2 (1) \\ \Rightarrow r &= \sqrt{169} = 13\end{aligned}\quad \left| \begin{array}{l} \frac{5}{12} = \frac{r \sin \phi}{r \cos \phi} \\ \frac{5}{12} = \tan \phi \\ \Rightarrow \phi = \tan^{-1} \frac{5}{12} \end{array} \right.$$

Now

$$\begin{aligned}12 \sin \theta + 5 \cos \theta &= r \cos \phi \sin \theta + r \sin \phi \cos \theta \\ &= r (\cos \phi \sin \theta + \sin \phi \cos \theta)\end{aligned}$$

$$= r \sin(\theta + \varphi) \quad \text{where } r = 13 \text{ and } \varphi = \tan^{-1} \frac{5}{12}$$

(ii) $3 \sin \theta - 4 \cos \theta$

Let $3 = r \cos \varphi$ and $-4 = r \sin \varphi$

Squaring and adding

$$\begin{aligned} (3)^2 + (-4)^2 &= r^2 \cos^2 \varphi + r^2 \sin^2 \varphi \\ \Rightarrow 9 + 16 &= r^2 (\cos^2 \varphi + \sin^2 \varphi) \\ \Rightarrow 25 &= r^2 (1) \\ \Rightarrow r &= \sqrt{25} = 5 \end{aligned}$$

$$\begin{aligned} \frac{-4}{3} &= \frac{r \sin \varphi}{r \cos \varphi} \\ \frac{-4}{3} &= \tan \varphi \\ \Rightarrow \varphi &= \tan^{-1} \left(-\frac{4}{3} \right) \end{aligned}$$

$$\begin{aligned} 3 \sin \theta - 4 \cos \theta &= r \cos \varphi \sin \theta + r \sin \varphi \cos \theta \\ &= r (\cos \varphi \sin \theta + \sin \varphi \cos \theta) \end{aligned}$$

$$= r \sin(\theta + \varphi) \quad \text{where } r = 5 \text{ and } \varphi = \tan^{-1} \left(-\frac{4}{3} \right)$$

(iii) $\sin \theta - \cos \theta$

Let $1 = r \cos \varphi$ and $-1 = r \sin \varphi$

Squaring and adding

$$\begin{aligned} (1)^2 + (-1)^2 &= r^2 \cos^2 \varphi + r^2 \sin^2 \varphi \\ \Rightarrow 1 + 1 &= r^2 (\cos^2 \varphi + \sin^2 \varphi) \\ \Rightarrow 2 &= r^2 (1) \\ \Rightarrow r &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \frac{-1}{1} &= \frac{r \sin \varphi}{r \cos \varphi} \\ -1 &= \tan \varphi \\ \Rightarrow \varphi &= \tan^{-1}(-1) \end{aligned}$$

Now

$$\begin{aligned} \sin \theta - \cos \theta &= r \cos \varphi \sin \theta + r \sin \varphi \cos \theta \\ &= r (\cos \varphi \sin \theta + \sin \varphi \cos \theta) \\ &= r \sin(\theta + \varphi) \quad \text{where } r = \sqrt{2} \text{ and } \varphi = \tan^{-1}(-1) \end{aligned}$$

(iv) $5 \sin \theta - 4 \cos \theta$

Let $5 = r \cos \varphi$ and $-4 = r \sin \varphi$

Squaring and adding

$$\begin{aligned} (5)^2 + (-4)^2 &= r^2 \cos^2 \varphi + r^2 \sin^2 \varphi \\ \Rightarrow 25 + 16 &= r^2 (\cos^2 \varphi + \sin^2 \varphi) \\ \Rightarrow 41 &= r^2 (1) \\ \Rightarrow r &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} \frac{-4}{5} &= \frac{r \sin \varphi}{r \cos \varphi} \\ \frac{-4}{5} &= \tan \varphi \\ \Rightarrow \varphi &= \tan^{-1} \left(-\frac{4}{5} \right) \end{aligned}$$

Now

$$\begin{aligned} 5 \sin \theta - 4 \cos \theta &= r \cos \varphi \sin \theta + r \sin \varphi \cos \theta \\ &= r (\cos \varphi \sin \theta + \sin \varphi \cos \theta) \end{aligned}$$

$$= r \sin(\theta + \varphi) \quad \text{where } r = \sqrt{41} \text{ and } \varphi = \tan^{-1}\left(-\frac{4}{5}\right)$$

(v) $\sin \theta + \cos \theta$ Let $1 = r \cos \varphi$ and $1 = r \sin \varphi$

Squaring and adding

$$(1)^2 + (1)^2 = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi$$

$$\Rightarrow 1+1 = r^2 (\cos^2 \varphi + \sin^2 \varphi)$$

$$\Rightarrow 2 = r^2 (1)$$

$$\Rightarrow r = \sqrt{2}$$

$$\begin{aligned} \frac{1}{1} &= \frac{r \sin \theta}{r \cos \theta} \\ 1 &= \tan \theta \\ \Rightarrow \varphi &= \tan^{-1}(1) \end{aligned}$$

Now $\sin \theta + \cos \theta = r \cos \varphi \sin \theta + r \sin \varphi \cos \theta$

$$= r(\cos \varphi \sin \theta + \sin \varphi \cos \theta)$$

$$= r \sin(\theta + \varphi)$$

where $r = \sqrt{2}$ and $\varphi = \tan^{-1}(1)$

(vi)

Do yourself as above