

**Question # 1**

Find the values of  $\sin 2\alpha$ ,  $\cos 2\alpha$  and  $\tan 2\alpha$  when:

$$(i) \quad \sin \alpha = \frac{12}{13} \quad (ii) \quad \cos \alpha = \frac{3}{5} \quad \text{where } 0 < \alpha < \frac{\pi}{2}$$

**Solution**

$$(i) \quad \sin \alpha = \frac{12}{13} ; \quad 0 < \alpha < \frac{\pi}{2}$$

$$\text{Since } \cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}$$

As  $\alpha$  is in the first quadrant so value of cos is +ive

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha}$$

$$= \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{25}{169}} \Rightarrow \cos \alpha = \frac{5}{13}$$

$$\text{and } \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{12/13}{5/13} \Rightarrow \tan \alpha = \frac{12}{5}$$

$$\text{Now } \sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$= 2 \left(\frac{12}{13}\right) \left(\frac{5}{13}\right) \Rightarrow \boxed{\sin 2\alpha = \frac{120}{169}}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \\ = \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} \Rightarrow \boxed{\cos 2\alpha = \frac{119}{169}}$$

$$\tan 2\alpha = \frac{2\alpha \tan \alpha}{1 - \tan^2 \alpha} \\ = \frac{2(12/5)}{1 - (12/5)^2} = \frac{24/5}{1 - 144/25} = \frac{24/5}{-119/25} = -\frac{24}{5} \cdot \frac{25}{119}$$

$$\Rightarrow \boxed{\tan 2\alpha = \frac{120}{119}}$$

$$(ii) \quad \cos \alpha = \frac{3}{5} ; \quad 0 < \alpha < \frac{\pi}{2}$$

**Hint:** First find  $\sin \alpha$  and  $\tan \alpha$  then solve as above

Prove the following identities (**Question 2 – 13**)

**Question # 2**

$$\cot \alpha - \tan \alpha = 2 \cot 2\alpha$$

**Solution**      L.H.S =  $\cot \alpha - \tan \alpha = \frac{\cos \alpha}{\sin \alpha} - \frac{\sin \alpha}{\cos \alpha} = \frac{\cos^2 \alpha - \sin^2 \alpha}{\sin \alpha \cos \alpha}$   
 $= \frac{2(\cos^2 \alpha - \sin^2 \alpha)}{2 \sin \alpha \cos \alpha} = \frac{2 \cos 2\alpha}{\sin 2\alpha} = 2 \cot 2\alpha = \text{R.H.S}$

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**Question # 3**

$$\frac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan \alpha$$

**Solution**

$$\begin{aligned} \text{L.H.S} &= \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \frac{2 \sin \alpha \cos \alpha}{2 \cos^2 \alpha} \\ &= \frac{\sin \alpha}{\cos \alpha} = \tan \alpha = \text{R.H.S} \end{aligned}$$


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$$\left| \begin{array}{l} \because \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \\ \therefore 2 \cos^2 \alpha = 1 + \cos 2\alpha \end{array} \right.$$

**Question # 4**

$$\frac{1 - \cos \alpha}{\sin \alpha} = \tan \frac{\alpha}{2}$$

**Solution**

$$\begin{aligned} \text{L.H.S} &= \frac{1 - \cos \alpha}{\sin \alpha} = \frac{2 \sin^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \\ &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \tan \frac{\alpha}{2} = \text{R.H.S} \end{aligned}$$


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$$\left| \begin{array}{l} \because \sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2} \\ \therefore 2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha \\ \sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \end{array} \right.$$

**Question # 5**

$$\frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} = \sec 2\alpha - \tan 2\alpha$$

**Solution**

$$\begin{aligned} \text{L.H.S} &= \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \\ &= \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \times \frac{\cos \alpha - \sin \alpha}{\cos \alpha - \sin \alpha} \\ &= \frac{(\cos \alpha - \sin \alpha)^2}{\cos^2 \alpha - \sin^2 \alpha} = \frac{\cos^2 \alpha + \sin^2 \alpha - 2 \sin \alpha \cos \alpha}{\cos 2\alpha} \\ &= \frac{1 - \sin 2\alpha}{\cos 2\alpha} = \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha} = \sec 2\alpha - \tan 2\alpha = \text{R.H.S} \end{aligned}$$


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**Question 6**

$$\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}$$

**Solution** L.H.S =  $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$

$$\begin{aligned} &= \sqrt{\frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}} \\ &= \sqrt{\frac{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}} = \text{R.H.S} \end{aligned}$$

$\therefore \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1$   
 $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

**Question # 7**

$$\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta} = \cot\frac{\theta}{2}$$

**Solution** L.H.S =  $\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta}$

$$\begin{aligned} &= \frac{\frac{1}{\sin\theta} + \frac{2}{\sin2\theta}}{\frac{1}{\cos\theta}} = \cos\theta \left( \frac{1}{\sin\theta} + \frac{2}{2\sin\theta\cos\theta} \right) \\ &= \cos\theta \left( \frac{1}{\sin\theta} + \frac{1}{\sin\theta\cos\theta} \right) = \cos\theta \left( \frac{\cos\theta + 1}{\sin\theta\cos\theta} \right) \\ &= \frac{\cos\theta + 1}{\sin\theta} = \frac{\frac{2\cos^2\frac{\theta}{2}}{2}}{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2}} = \cot\frac{\theta}{2} = \text{R.H.S} \end{aligned}$$

**Question # 8**

$$1 + \tan\alpha \tan 2\alpha = \sec 2\alpha$$

**Solution** L.H.S =  $1 + \tan\alpha \tan 2\alpha = 1 + \left( \frac{\sin\alpha}{\cos\alpha} \right) \left( \frac{\sin 2\alpha}{\cos 2\alpha} \right)$

$$\begin{aligned} &= \frac{\cos\alpha \cos 2\alpha + \sin\alpha \sin 2\alpha}{\cos\alpha \cos 2\alpha} = \frac{\cos(2\alpha - \alpha)}{\cos\alpha \cos 2\alpha} \\ &= \frac{\cos\alpha}{\cos\alpha \cos 2\alpha} = \frac{1}{\cos 2\alpha} = \sec 2\alpha = \text{R.H.S} \end{aligned}$$

**Question # 9**

$$\frac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta} = \tan 2\theta \tan \theta$$

Solution L.H.S =  $\frac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta}$

$$= \frac{2\sin\theta\sin 2\theta}{\cos\theta+4\cos^3\theta-3\cos\theta}$$

$$= \frac{2\sin\theta\sin 2\theta}{4\cos^3\theta-2\cos\theta} = \frac{2\sin\theta\sin 2\theta}{2\cos\theta(2\cos^2\theta-1)}$$

$$= \frac{\sin\theta\sin 2\theta}{\cos\theta\cos 2\theta} = \tan\theta \tan 2\theta = \tan 2\theta \tan\theta = \text{R.H.S}$$

$$\begin{aligned}\because \cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \therefore \cos^2\theta &= \frac{1+\cos 2\theta}{2} \\ \therefore 2\cos^2\theta - 1 &= \cos 2\theta\end{aligned}$$

**Question # 10**

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$$

Solution L.H.S =  $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta \cos\theta - \cos 3\theta \sin\theta}{\sin\theta \cos\theta}$

$$= \frac{\sin(3\theta - \theta)}{\sin\theta \cos\theta} = \frac{\sin 2\theta}{\sin\theta \cos\theta} = \frac{2\sin\theta \cos\theta}{\sin\theta \cos\theta} = 2 = \text{R.H.S}$$

**Question # 11**

$$\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$$

Solution L.H.S =  $\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = \frac{\cos 3\theta \sin\theta + \sin 3\theta \cos\theta}{\sin\theta \cos\theta}$

$$= \frac{\sin(\theta + 3\theta)}{\sin\theta \cos\theta} = \frac{\sin 4\theta}{\sin\theta \cos\theta} = \frac{2\sin 2\theta \cos 2\theta}{\sin\theta \cos\theta}$$

$$= \frac{2(2\sin\theta \cos\theta) \cos 2\theta}{\sin\theta \cos\theta} = 4\cos 2\theta = \text{R.H.S}$$

**Question # 12**

$$\frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}} = \sec\theta$$

Solution L.H.S =  $\frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}}$

$$= \frac{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}}{\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}} = \frac{\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}{\frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$$

$$=\frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{1}{\cos \theta} = \sec \theta = \text{R.H.S}$$


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**Question # 13**

$$\frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} = 2 \cot 2\theta$$

**Solution**      L.H.S =  $\frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} = \frac{\sin 3\theta \sin \theta + \cos 3\theta \cos \theta}{\sin \theta \cos \theta}$   
 $= \frac{\cos(3\theta - \theta)}{\sin \theta \cos \theta} = \frac{\cos 2\theta}{\sin \theta \cos \theta} = \frac{2 \cos 2\theta}{2 \sin \theta \cos \theta}$   
 $= \frac{2 \cos 2\theta}{\sin 2\theta} = 2 \cot 2\theta = \text{R.H.S}$

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**Question # 14**

Reduce  $\sin^4 \theta$  to an expression involving only functions of multiples of  $\theta$  raised to the first power.

**Solution**       $\sin^4 \theta = (\sin^2 \theta)^2 = \left(\frac{1 - \cos 2\theta}{2}\right)^2$   
 $= \frac{1 - 2 \cos 2\theta + \cos^2 2\theta}{4} = \frac{1}{4}(1 - 2 \cos 2\theta + \cos^2 2\theta)$   
 $= \frac{1}{4}\left(1 - 2 \cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) = \frac{1}{4}\left(\frac{2 - 4 \cos 2\theta + 1 + \cos 4\theta}{2}\right)$   
 $= \frac{1}{8}(3 - 4 \cos 2\theta + \cos 4\theta)$

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**Question # 15**

Find the values of  $\sin \theta$  and  $\cos \theta$ , without using table or calculator, when  $\theta$

- (i)  $18^\circ$       (ii)  $36^\circ$       (iii)  $54^\circ$       (iv)  $72^\circ$

**Solution**

(i) Let  $\theta = 18^\circ \Rightarrow 5\theta = 90^\circ \Rightarrow 3\theta + 2\theta = 90^\circ \Rightarrow 2\theta = 90^\circ - 3\theta$   
 $\sin 2\theta = \sin(90^\circ - 3\theta)$   
 $\Rightarrow \sin 2\theta = \cos 3\theta$   
 $\Rightarrow 2 \sin \theta \cos \theta = 4 \cos^3 \theta - 3 \cos \theta$   
 $\Rightarrow 2 \sin \theta = 4 \cos^2 \theta - 3$        $\div \text{ing by } \cos \theta$        $\left| \begin{array}{l} \because \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \\ \therefore \sin 2\theta = 2 \sin \theta \cos \theta \end{array} \right.$   
 $\Rightarrow 2 \sin \theta = 4(1 - \sin^2 \theta) - 3$   
 $\Rightarrow 2 \sin \theta = 4 - 4 \sin^2 \theta - 3 \Rightarrow 2 \sin \theta = 1 - 4 \sin^2 \theta$   
 $\Rightarrow 4 \sin^2 \theta + 2 \sin \theta - 1 = 0$

This is quadratic in  $\sin \theta$  with  $a=4$ ,  $b=1$  and  $c=-1$

$$\begin{aligned}\sin \theta &= \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2(4)} = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm \sqrt{20}}{8} \\ &= \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}\end{aligned}$$

Since  $\theta = 18^\circ$  lies in the first quadrant so value of sin can not be negative therefore

$$\sin \theta = \frac{-1 + \sqrt{5}}{4} \Rightarrow \boxed{\sin 18^\circ = \frac{\sqrt{5}-1}{4}} \quad \because \theta = 18^\circ$$

Now

$$\begin{aligned}\cos^2 18^\circ &= 1 - \sin^2 18^\circ \Rightarrow \cos^2 18^\circ = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 \\ \Rightarrow \cos^2 18^\circ &= 1 - \frac{(\sqrt{5})^2 - 2\sqrt{5} + 1}{16} = 1 - \frac{5 - 2\sqrt{5} + 1}{16} \\ &= 1 - \frac{6 - 2\sqrt{5}}{16} = \frac{16 - 6 + \sqrt{5}}{16} = \frac{10 + 2\sqrt{5}}{16} \\ \Rightarrow \cos 18^\circ &= \sqrt{\frac{10 + 2\sqrt{5}}{16}} \Rightarrow \boxed{\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}}\end{aligned}$$

$$\begin{aligned}\text{(ii) Since } \cos^2 \theta &= \frac{1 + \cos 2\theta}{2} \\ \Rightarrow \cos 2\theta &= 2\cos^2 \theta - 1 \\ \Rightarrow \cos 2(18) &= 2\cos^2(18) - 1 \\ \Rightarrow \cos 36 &= 2\left(\frac{\sqrt{10 + 2\sqrt{5}}}{4}\right)^2 - 1 \\ &= 2\left(\frac{10 + 2\sqrt{5}}{16}\right) - 1 = \frac{10 + 2\sqrt{5}}{8} - 1 \\ &= \frac{10 + 2\sqrt{5} - 8}{8} = \frac{2 + 2\sqrt{5}}{8} \Rightarrow \boxed{\cos 36^\circ = \frac{1 + \sqrt{5}}{4}}\end{aligned}$$

Now  $\sin^2 36 = 1 - \cos^2 36$

$$\begin{aligned}&= 1 - \left(\frac{1 + \sqrt{5}}{2}\right)^2 = 1 - \frac{1 + 2\sqrt{5} + (\sqrt{5})^2}{16} \\ &= 1 - \frac{1 + 2\sqrt{5} + 5}{16} = 1 - \frac{6 + 2\sqrt{5}}{16} \\ &= \frac{16 - 6 - 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16}\end{aligned}$$

$$\Rightarrow \sin 36^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}} \Rightarrow \boxed{\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

(iii) Now  $\sin(90-36) = \cos 36^\circ \quad \therefore \sin(90-\theta) = \cos \theta$

$$\Rightarrow \sin 54^\circ = \cos 36^\circ \Rightarrow \boxed{\sin 54^\circ = \frac{1+\sqrt{5}}{4}}$$

And  $\cos(90-36) = \sin 36^\circ$

$$\Rightarrow \cos 54^\circ = \sin 36^\circ \Rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

(iv) Now  $\sin(90-18) = \cos 18^\circ \quad \therefore \sin(90-\theta) = \cos \theta$

$$\Rightarrow \sin 72^\circ = \cos 18^\circ \Rightarrow \boxed{\sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

and  $\cos(90-18) = \sin 18^\circ$

$$\Rightarrow \cos 72^\circ = \sin 18^\circ \Rightarrow \boxed{\cos 72^\circ = \frac{\sqrt{5}-1}{4}}$$

### Alternative Method for Q # 15 (iii)

Let  $\theta = 54^\circ \Rightarrow 5\theta = 270^\circ \Rightarrow 3\theta + 2\theta = 270^\circ \Rightarrow 2\theta = 270^\circ - 3\theta$

$$\sin 2\theta = \sin(270^\circ - 3\theta)$$

$$\Rightarrow \sin 2\theta = \sin(3(90^\circ) - 3\theta)$$

$$\Rightarrow \sin 2\theta = -\cos 3\theta$$

$$\Rightarrow 2\sin \theta \cos \theta = -(4\cos^3 \theta - 3\cos \theta)$$

$$\Rightarrow 2\sin \theta \cos \theta = -4\cos^3 \theta + 3\cos \theta$$

$$\Rightarrow 2\sin \theta = -4\cos^2 \theta + 3 \quad \text{dividing by } \cos \theta$$

$$\Rightarrow 2\sin \theta = -4(1 - \sin^2 \theta) + 3$$

$$\Rightarrow 2\sin \theta = -4 + 4\sin^2 \theta + 3 \Rightarrow 2\sin \theta = 4\sin^2 \theta - 1$$

$$\Rightarrow 4\sin^2 \theta - 2\sin \theta - 1 = 0$$

$$\because \cos 3\theta = 4\cos^3 \theta - 3\cos \theta$$

$$\therefore \sin 2\theta = 2\sin \theta \cos \theta$$

This is quadratic in  $\sin \theta$  with  $a = 4$ ,  $b = 1$  and  $c = -1$

$$\begin{aligned} \sin \theta &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)} = \frac{2 \pm \sqrt{4+16}}{8} = \frac{2 \pm \sqrt{20}}{8} \\ &= \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4} \end{aligned}$$

Since  $\theta = 54^\circ$  lies in the first quadrant so value of sin can not be negative therefore

$$\sin \theta = \frac{1+\sqrt{5}}{4} \Rightarrow \boxed{\sin 54^\circ = \frac{1+\sqrt{5}}{4}} \quad \therefore \theta = 54^\circ$$

Now

$$\begin{aligned} \cos^2 54^\circ &= 1 - \sin^2 54^\circ \Rightarrow \cos^2 54^\circ = 1 - \left( \frac{1+\sqrt{5}}{4} \right)^2 \\ \Rightarrow \cos^2 54^\circ &= 1 - \frac{(\sqrt{5})^2 + 2\sqrt{5} + 1}{16} = 1 - \frac{5 + 2\sqrt{5} + 1}{16} \\ &= 1 - \frac{6 + 2\sqrt{5}}{16} = \frac{16 - 6 - \sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16} \\ \Rightarrow \cos 54^\circ &= \sqrt{\frac{10 - 2\sqrt{5}}{16}} \Rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}} \end{aligned}$$


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