

Chapter 11

Exercise 11.1

Domains and Ranges of Sine and Cosine Functions

Let us consider a unit circle with centre at origin O.

Let $P(x, y)$ be any point on the circle such that $\angle xOP = \theta$ is in standard position. Then

$$\sin \theta = \frac{y}{1} \Rightarrow \sin \theta = y$$

$$\cos \theta = \frac{x}{1} \Rightarrow \cos \theta = x$$

\Rightarrow Corresponding to any real number θ , there is one and only one value of x and y i.e., one and only one value for each $\sin \theta$ and $\cos \theta$.

Hence $\sin \theta$ and $\cos \theta$ are functions of θ .

\therefore $\sin \theta$ and $\cos \theta$ are defined for all $\theta \in \mathbb{R}$, the set of real numbers.

\therefore Domain of $\sin \theta = \mathbb{R}$

Domain of $\cos \theta = \mathbb{R}$

To find the range, we have

Since $P(x, y)$ is a point on the unit circle with centre at O.

$$\therefore -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1$$

$$\Rightarrow -1 \leq \cos \theta \leq 1 \text{ and } -1 \leq \sin \theta \leq 1$$

Domains and Ranges of Tangent and Cotangent Functions:

From figure ; $\tan \theta = \frac{y}{x}$, $x \neq 0$

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\Rightarrow terminal side \overrightarrow{OP} should not coincide with oy or oy' (i.e., y-axis)

$$\Rightarrow \theta \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$

$$\Rightarrow \theta \neq (2n+1) \frac{\pi}{2}, n \in \mathbb{Z}$$

\therefore Domain of $\tan \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq (2n+1) \frac{\pi}{2}$, $n \in \mathbb{Z}$

and Range of $\tan \theta = \mathbb{R}$

$$\text{Now } \cot \theta = \frac{1}{\tan \theta} = \frac{x}{y}, y \neq 0 \quad [2]$$

\Rightarrow terminal side \overrightarrow{OP} should not coincide with OX or OX' (i.e., X -axis).

$\Rightarrow \theta \neq 0, \pm \pi, \pm 2\pi, \dots$

$\Rightarrow \theta \neq n\pi, n \in \mathbb{Z}$

\therefore domain of $\cot \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq n\pi, n \in \mathbb{Z}$.

and range of $\cot \theta$ is \mathbb{R} .

Domains and Ranges of Secant and Cosecant Functions.

$$\text{From fig. } \sec \theta = \frac{1}{\cos \theta} = \frac{1}{x}, x \neq 0$$

\Rightarrow terminal side \overrightarrow{OP} should not coincide with OY or OY' (i.e., Y -axis)

$\Rightarrow \theta \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

$\Rightarrow \theta \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

\therefore Domain of $\sec \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$.

As $\sec \theta$ attains all real values except those between -1 and 1.

\therefore Range of $\sec \theta = \mathbb{R} - \{x | -1 < x < 1\}$

$$\text{Now } \csc \theta = \frac{1}{\sin \theta} = \frac{1}{y}, y \neq 0$$

\Rightarrow terminal side \overrightarrow{OP} should not coincide with OX or OX' (i.e., X -axis)

$\Rightarrow \theta \neq 0, \pm \pi, \pm 2\pi, \dots$

$\Rightarrow \theta \neq n\pi, n \in \mathbb{Z}$

\therefore Domain of $\csc \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq n\pi, n \in \mathbb{Z}$

As $\csc \theta$ attains all real values except those between -1 and 1.

\therefore Range of $\csc \theta = \mathbb{R} - \{x | -1 < x < 1\}$

Now summarizing the above results in the form of a table as:

Function	Domain	Range
$y = \sin x$	\mathbb{R}	$-1 \leq y \leq 1$
$y = \cos x$	\mathbb{R}	$-1 \leq y \leq 1$
$y = \tan x$	$x \in \mathbb{R} \text{ but } x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$	\mathbb{R}
$y = \cot x$	$x \in \mathbb{R} \text{ but } x \neq n\pi, n \in \mathbb{Z}$	\mathbb{R}
$y = \sec x$	$x \in \mathbb{R} \text{ but } x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$	$\mathbb{R} - \{x -1 < x < 1\}$
$y = \csc x$	$x \in \mathbb{R} \text{ but } x \neq n\pi, n \in \mathbb{Z}$	$\mathbb{R} - \{x -1 < x < 1\}$

Periodic Function

A function f is said to be periodic if for every x belonging to its domain D , there exists a positive number p such that $x+p \in D$ and $f(x+p) = f(x)$.

If p is the least positive number satisfying these conditions, then it is called the period of f .

Periodicity: All the six trigonometric functions repeat their values for each increase or decrease of 2π in θ . This behaviour of trigonometric functions is called periodicity.

Theorem

Sine is a periodic function and its period is 2π .

Proof: Let p be the period of sine. Then

$$\sin(\theta+p) = \sin \theta \quad \forall \theta \in \mathbb{R} \quad \text{①}$$

putting $\theta=0$ in ①, we get

$$\sin(\theta+p) = \sin \theta \Rightarrow \sin p = 0$$

$$\Rightarrow p = \sin^{-1}(0)$$

$$\Rightarrow p = 0, \pi, 2\pi, \dots$$

i) If $p=\pi$, then from ①

$$\sin(\theta+\pi) = \sin \theta$$

$$\Rightarrow -\sin \theta = \sin \theta \quad (\text{not true})$$

$$\therefore \pi \text{ is not the period of } \sin \theta$$

ii) If $p=2\pi$, then from ①

$$\sin(\theta+2\pi) = \sin \theta$$

$$\Rightarrow \sin \theta = \sin \theta \quad (\text{true}) \quad \because \sin(\theta+2\pi) = \sin \theta$$

$\therefore 2\pi$ is the period of $\sin \theta$

Theorem: Tangent is a periodic function and its period is π .

Proof: Let p be the period of \tan . Then

$$\tan(\theta+p) = \tan \theta \quad \forall \theta \in \mathbb{R} \quad \text{②}$$

Putting $\theta=0$ in ②, we get

$$\tan(\theta+p) = \tan \theta$$

$$\Rightarrow \tan p = 0$$

$$\Rightarrow p = 0, \pi, 2\pi, 3\pi, \dots$$

$b=0$ can't be the period of $\tan\theta$ $\because b=0$ is not positive.

If $b=\pi$, then from ①

$$\tan(\theta+\pi) = \tan\theta$$

$$\Rightarrow \tan\theta = \tan\theta \text{ (true)}$$

$\therefore \pi$ is the period of $\tan\theta$

\because it is the least +ve number for which $\tan(\theta+\pi) = \tan\theta$.

Similarly we can prove that

i) 2π is the period of $\cos\theta$

ii) 2π is the period of $\csc\theta$

iii) 2π is the period of $\sec\theta$

iv) π is the period of $\cot\theta$.

* EXERCISE 11.1 *

Find the periods of the following functions.

$$1) \sin 3x = \sin(3x+2\pi) \\ = \sin 3(x+\frac{2\pi}{3})$$

$$\therefore \text{period of } \sin 3x = \frac{2\pi}{3} \text{ Ans.}$$

$$2) \cos 2x = \cos(2x+2\pi) \\ = \cos 2(x+\pi)$$

$$\therefore \text{period of } \cos 2x = \pi \text{ Ans.}$$

$$3) \tan 4x = \tan(4x+\pi) \\ = \tan 4(x+\frac{\pi}{4})$$

$$\therefore \text{period of } \tan 4x = \frac{\pi}{4} \text{ Ans.}$$

$$④ \cot \frac{x}{2} = \cot(\frac{x}{2} + \pi) \\ = \cot \frac{1}{2}(x+2\pi)$$

$$\therefore \text{period of } \cot \frac{x}{2} = 2\pi \text{ Ans.}$$

$$⑤ \sin \frac{x}{3} = \sin(\frac{x}{3} + 2\pi) \\ = \sin \frac{1}{3}(x+6\pi)$$

$$\therefore \text{period of } \sin \frac{x}{3} = 6\pi \text{ Ans.}$$

$$⑥ \csc \frac{x}{4} = \csc(\frac{x}{4} + 2\pi) \\ = \csc \frac{1}{4}(x+8\pi)$$

$$\therefore \text{period of } \csc \frac{x}{4} = 8\pi \text{ Ans.}$$

$$⑦ \sin \frac{x}{5} = \sin(\frac{x}{5} + 2\pi) \\ = \frac{1}{5} \sin(x+10\pi)$$

$$\therefore \text{period of } \sin \frac{x}{5} = 10\pi \text{ Ans.}$$

$$⑧ \cos \frac{x}{6} = \cos(\frac{x}{6} + 2\pi) \\ = \cos \frac{1}{6}(x+12\pi)$$

$$\therefore \text{period of } \cos \frac{x}{6} = 12\pi$$

$$⑨ \tan \frac{x}{7} = \tan(\frac{x}{7} + \pi) \\ = \tan \frac{1}{7}(x+7\pi)$$

$$\therefore \text{period of } \tan \frac{x}{7} = 7\pi \text{ Ans.}$$

$$⑩ \cot 8x = \cot(8x+\pi) \\ = \cot 8(x+\frac{\pi}{8})$$

$$\therefore \text{period of } \cot 8x = \frac{\pi}{8} \text{ Ans.}$$

$$⑪ \sec 9x = \sec(9x+2\pi) \\ = \sec 9(x+\frac{2\pi}{9})$$

$$\therefore \text{period of } \sec 9x = \frac{2\pi}{9} \text{ Ans.}$$

(12) $\operatorname{Cosec} 10x$

$$= \operatorname{Cosec}(10x + 2\pi)$$

$$= \operatorname{Cosec} 10(x + \frac{2\pi}{10})$$

$$= \operatorname{Cosec} 10(x + \frac{\pi}{5})$$

\therefore period of $\operatorname{Cosec} 10x = \frac{\pi}{5}$ Ans.

(13) $3\sin x = 3 \sin(x + 2\pi)$

\therefore period of $3\sin x = 2\pi$ Ans.

(14) $2\cos x = 2 \cos(x + 2\pi)$

\therefore period of $2\cos x = 2\pi$ Ans.

(15) $3 \cos \frac{x}{5} = 3 \cos(\frac{x}{5} + 2\pi)$

$$= 3 \cos \frac{1}{5}(x + 10\pi)$$

\therefore period of $3 \cos \frac{x}{5} = 10\pi$ Ans.

EXERCISE 11.2

1. i) $y = -\sin x ; x \in [-2\pi, 2\pi]$

x	-360°	-330°	-300°	-270°	-240°	-210°	-180°	-150°	-120°	-90°	-60°	-30°	0°
y	0	-0.5	-0.9	-1	-0.9	-0.5	0	0.5	0.9	1	0.9	0.5	0
x	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°	
y	-0.5	-0.9	-1	-0.9	-0.5	0	0.5	0.9	1	0.9	0.5	0	

Scale: One big square along x-axis = 100°

One big square along y-axis = 1 unit.

