

Chapter 13

Exercise 13.1

1. Evaluate without using tables / calculator:

$$\begin{array}{lll} \text{i)} \quad \sin^{-1}(1) & \text{ii)} \quad \sin^{-1}(-1) & \text{iii)} \quad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ \text{iv)} \quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) & \text{v)} \quad \cos^{-1}\left(\frac{1}{2}\right) & \text{vi)} \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ \text{vii)} \quad \cot^{-1}(-1) & \text{viii)} \quad \cosec^{-1}\left(\frac{-2}{\sqrt{3}}\right) & \text{ix)} \quad \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) \end{array}$$

2. Without using table/ Calculator show that:

$$\begin{array}{lll} \text{i)} \quad \tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13} & \text{ii)} \quad 2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25} \\ \text{iii)} \quad \cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3} \end{array}$$

3) Find the value of each expression:

$$\begin{array}{lll} \text{i)} \quad \cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) & \text{ii)} \quad \sec\left(\cos^{-1}\frac{1}{2}\right) & \text{iii)} \quad \tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) \\ \text{iv)} \quad \csc\left(\tan^{-1}(-1)\right) & \text{v)} \quad \sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) & \text{vi)} \quad \tan\left(\tan^{-1}(-1)\right) \\ \text{vii)} \quad \sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right) & \text{viii)} \quad \tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) & \text{ix)} \quad \sin\left(\tan^{-1}(-1)\right) \end{array}$$

Solution Are Given Below

Solution # 1

$$\begin{aligned} \text{(i)} \quad \text{Suppose } y &= \sin^{-1}(1) & \text{where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \Rightarrow \sin y &= 1 \end{aligned}$$

$$\Rightarrow y = \frac{\pi}{2} \quad \because \sin\left(\frac{\pi}{2}\right) = 1 \text{ Answer}$$

$$\begin{aligned} \text{(ii)} \quad \text{Suppose } y &= \sin^{-1}(-1) & \text{where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \Rightarrow \sin y &= -1 \end{aligned}$$

$$\Rightarrow y = -\frac{\pi}{2} \quad \because \sin\left(-\frac{\pi}{2}\right) = -1 \text{ Answer}$$

(iii) Suppose $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ where $y \in [0, \pi]$

$$\Rightarrow \cos y = \frac{\sqrt{3}}{2}$$

$$\Rightarrow y = \frac{\pi}{6} \quad \because \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ Answer}$$

(iv) Suppose $y = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \tan y = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow y = -\frac{\pi}{6} \quad \because \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}} \text{ Answer}$$

(v) Suppose $y = \cos^{-1}\left(\frac{1}{2}\right)$ where $y \in [0, \pi]$

$$\Rightarrow \cos y = \frac{1}{2}$$

$$\Rightarrow y = \frac{\pi}{3} \quad \because \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}. \text{ Answer}$$

(vi) Suppose $y = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \tan y = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y = \frac{\pi}{6} \quad \because \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}. \text{ Answer}$$

(vii) Suppose $y = \cot^{-1}(-1)$ where $y \in]0, \pi[$

$$\Rightarrow \cot y = -1$$

$$\Rightarrow \frac{1}{\cot y} = \frac{1}{-1}$$

$$\Rightarrow \tan y = -1$$

$$\Rightarrow y = \frac{3\pi}{4} \quad \because \tan\left(\frac{3\pi}{4}\right) = -1 \text{ Answer}$$

(viii) Suppose $y = \operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$

$$\Rightarrow \operatorname{cosec} y = -\frac{2}{\sqrt{3}}$$

$$\begin{aligned}
 &\Rightarrow \frac{1}{\cosec y} = \frac{1}{-\frac{\sqrt{3}}{2}} \\
 &\Rightarrow \sin y = -\frac{\sqrt{3}}{2} \\
 &\Rightarrow y = -\frac{\pi}{3} \quad \because \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \text{ Answer} \\
 (\text{ix}) \quad &\text{Suppose } y = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) \quad \text{where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
 &\Rightarrow \sin y = -\frac{1}{\sqrt{2}} \\
 &\Rightarrow y = -\frac{\pi}{4} \quad \because \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}
 \end{aligned}$$

Solution # 2

$$(i) \quad \text{Suppose } \alpha = \sin^{-1} \frac{5}{13} \quad \dots \quad (i) \quad \text{where } \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \sin \alpha = \frac{5}{13}$$

$$\text{Now } \cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}$$

Since $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ therefore \cos is +ive.

$$\begin{aligned}
 \cos \alpha &= +\sqrt{1 - \sin^2 \alpha} \\
 &= \sqrt{1 - \left(\frac{5}{13}\right)^2} = \sqrt{1 - \frac{25}{169}} \\
 &= \sqrt{\frac{144}{169}} = \frac{12}{13}
 \end{aligned}$$

$$\text{Now } \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

$$\Rightarrow \alpha = \tan^{-1} \frac{5}{12} \quad \dots \quad (ii)$$

From (i) and (ii)

$$\tan^{-1} \frac{5}{12} = \sin^{-1} \frac{5}{13} \quad \text{Proved}$$

$$(ii) \quad \text{Suppose } \alpha = 2 \cos^{-1} \frac{4}{5} \quad \dots \quad (i) \quad \text{where } \frac{\alpha}{2} \in [0, \pi]$$

$$\Rightarrow \frac{\alpha}{2} = \cos^{-1} \frac{4}{5} \Rightarrow \cos \frac{\alpha}{2} = \frac{4}{5}$$

$$\text{Now } \sin \frac{\alpha}{2} = \pm \sqrt{1 - \cos^2 \frac{\alpha}{2}}$$

Since $\frac{\alpha}{2} \in [0, \pi]$ therefore sin is +ive.

$$\begin{aligned}\sin \frac{\alpha}{2} &= +\sqrt{1 - \cos^2 \frac{\alpha}{2}} \\ &= \sqrt{1 - \frac{16}{25}} = \sqrt{1 - \frac{16}{25}} \\ &= \sqrt{\frac{9}{25}} = \frac{3}{5}\end{aligned}$$

$$\text{Now } \sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \Rightarrow \sin \alpha = 2 \left(\frac{3}{5}\right) \left(\frac{4}{5}\right)$$

$$\Rightarrow \sin \alpha = \frac{24}{25} \Rightarrow \alpha = \sin^{-1} \frac{24}{25} \dots \dots \dots \text{(ii)}$$

From (i) and (ii)

$$2 \cos^{-1} \frac{4}{5} = \sin^{-1} \frac{24}{25} \quad \text{Prove}$$

$$\text{(iii)} \quad \text{Suppose } \alpha = \cos^{-1} \frac{4}{5} \dots \dots \dots \text{(i)} \quad \text{where } \alpha \in [0, \pi]$$

$$\Rightarrow \cos \alpha = \frac{4}{5}$$

$$\text{Now } \sin \alpha = \pm \sqrt{1 - \cos^2 \alpha}$$

Since $\alpha \in [0, \pi]$ therefore sin is +ive.

$$\begin{aligned}\sin \alpha &= +\sqrt{1 - \cos^2 \alpha} \\ &= \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}.\end{aligned}$$

$$\text{Now } \cot \alpha = \frac{\cos \alpha}{\sin \alpha} = \frac{\cancel{4}/5}{\cancel{3}/5} = \frac{4}{3}$$

$$\Rightarrow \alpha = \cot^{-1} \frac{4}{3} \dots \dots \dots \text{(ii)}$$

From (i) and (ii)

$$\cos^{-1} \frac{4}{5} = \cot^{-1} \frac{4}{3} \quad \text{Proved}$$

Solution # 3

$$\begin{aligned}\text{(i)} \quad \text{Suppose } y &= \sin^{-1} \frac{1}{\sqrt{2}} \quad \text{where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \Rightarrow \sin y &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\Rightarrow y = \frac{\pi}{4} \quad \because \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\text{Now } \cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) = \cos y = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{Answer}$$

□

$$(ii) \text{ Suppose } y = \cos^{-1}\frac{1}{2} \quad \text{where } y \in [0, \pi]$$

$$\Rightarrow \cos y = \frac{1}{2}$$

$$\Rightarrow y = \frac{\pi}{3} \quad \because \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\text{Now } \sec\left(\cos^{-1}\frac{1}{2}\right) = \sec y = \frac{1}{\cos y} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2 \quad \text{Answer}$$

(iii) *Do yourself*

□

$$(iv) \text{ Suppose } y = \tan^{-1}(-1) \quad \text{where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \tan y = -1$$

$$\Rightarrow y = -\frac{\pi}{4} \quad \because \tan\left(-\frac{\pi}{4}\right) = -1$$

$$\text{Now } \operatorname{cosec}\left(\tan^{-1}(-1)\right) = \operatorname{cosec} y = \frac{1}{\sin y} = \frac{1}{\sin\left(-\frac{\pi}{4}\right)} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2} \quad \text{Answer}$$

(v) *Do yourself*

$$(vi) \text{ Suppose } y = \tan^{-1}(-1) \quad \text{where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \tan y = -1$$

$$\Rightarrow y = -\frac{\pi}{4} \quad \because \tan\left(-\frac{\pi}{4}\right) = -1$$

$$\text{Now } \tan\left(\tan^{-1}(-1)\right) = \tan y = \tan\left(-\frac{\pi}{4}\right) = -1 \quad \text{Answer}$$

$$(vii) \text{ Suppose } y = \sin^{-1}\frac{1}{2} \quad \text{where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \sin y = \frac{1}{2}$$

$$\Rightarrow y = \frac{\pi}{6} \quad \because \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\text{Now } \sin\left(\sin^{-1}\frac{1}{2}\right) = \sin y = \sin\frac{\pi}{6} = \frac{1}{2} \quad \text{Answer}$$

□

(viii) Suppose $y = \sin^{-1}\left(-\frac{1}{2}\right)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \sin y = -\frac{1}{2}$$

$$\Rightarrow y = -\frac{\pi}{6} \quad \because \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$\text{Now } \tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan y = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}} \quad \text{Answer}$$

□

(ix) *Do yourself*
