

## Chapter 4

## Exercise 4.4

{ Question # 1

(i) Let  $x$  be a cube root of 8 then

$$\begin{aligned}x &= (8)^{\frac{1}{3}} \Rightarrow x^3 = 8 \\ \Rightarrow x^3 - 8 &= 0 \Rightarrow (x)^3 - (2)^3 = 0 \\ \Rightarrow (x-2)(x^2 + 2x + 4) &= 0 \\ \Rightarrow x-2 &= 0 \quad \text{or} \quad x^2 + 2x + 4 = 0 \\ \Rightarrow x &= 2 \quad \text{or} \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm \sqrt{-12}}{2} \\ &= \frac{-2 \pm 2\sqrt{-3}}{2} = 2\left(\frac{-1 \pm \sqrt{-3}}{2}\right) \\ \Rightarrow x &= 2\left(\frac{-1 + \sqrt{-3}}{2}\right) \quad \text{or} \quad x = 2\left(\frac{-1 - \sqrt{-3}}{2}\right) \\ \Rightarrow x &= 2\omega \quad \text{or} \quad x = 2\omega^2\end{aligned}$$

Review:

$$\omega = \frac{-1 + \sqrt{-3}}{2}$$
$$\omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

Hence cube root of 8 are  $2, 2\omega$  and  $2\omega^2$ .

(ii) Hint

Considering  $x$  as a cube root of  $-8$  and Solving as above you will get the following values of  $x$

$$\begin{aligned}x &= -2, \quad x = \frac{2 + 2\sqrt{-3}}{2}, \quad x = \frac{2 - 2\sqrt{-3}}{2} \\ \Rightarrow x &= -2\left(\frac{-1 - \sqrt{-3}}{2}\right), \quad x = -2\left(\frac{-1 + \sqrt{-3}}{2}\right) \\ \Rightarrow x &= -2\omega^2, \quad x = -2\omega\end{aligned}$$

Hence cube root of  $-8$  are  $-2, -2\omega$  and  $-2\omega^2$ .

(iii) Do yourself as (iv) below.

(iv) Let  $x$  be a cube root of  $-27$  then

$$\begin{aligned}x &= (-27)^{\frac{1}{3}} \Rightarrow x^3 = -27 \\ \Rightarrow x^3 + 27 &= 0 \Rightarrow (x)^3 + (3)^3 = 0 \\ \Rightarrow (x+3)(x^2 - 3x + 9) &= 0 \\ \Rightarrow x+3 &= 0 \quad \text{or} \quad x^2 - 3x + 9 = 0 \\ \Rightarrow x &= -3 \quad \text{or} \quad x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)} \\ &= \frac{3 \pm \sqrt{9-36}}{2} = \frac{3 \pm \sqrt{-27}}{2} = \frac{3 \pm 3\sqrt{-3}}{2} \\ \Rightarrow x &= \frac{3 + 3\sqrt{-3}}{2} \quad \text{or} \quad x = \frac{3 - 3\sqrt{-3}}{2}\end{aligned}$$

$$\Rightarrow x = -3 \left( \frac{-1 - \sqrt{-3}}{2} \right) \quad \text{or} \quad x = -3 \left( \frac{-1 + \sqrt{-3}}{2} \right)$$

$$\Rightarrow x = -3\omega^2 \quad \text{or} \quad x = -3\omega$$

Hence cube root of  $-27$  are  $-3$ ,  $-3\omega$  and  $-3\omega^2$ .

(v) Let  $x$  be a cube root of  $64$  then

$$\begin{aligned} x = (64)^{\frac{1}{3}} &\Rightarrow x^3 = 64 \\ \Rightarrow x^3 - 64 = 0 &\Rightarrow (x)^3 - (4)^3 = 0 \\ \Rightarrow (x - 4)(x^2 + 4x + 16) &= 0 \\ \Rightarrow x - 4 = 0 &\quad \text{or} \quad x^2 + 4x + 16 = 0 \\ \Rightarrow x = 4 &\quad \text{or} \quad x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{16 - 64}}{2} = \frac{-4 \pm \sqrt{-48}}{2} \\ &= \frac{-4 \pm 4\sqrt{-3}}{2} \quad \because 48 = 16 \times 3 \\ \Rightarrow x = \frac{-4 + 4\sqrt{-3}}{2} &\quad \text{or} \quad x = \frac{-4 - 4\sqrt{-3}}{2} \\ \Rightarrow x = 4 \left( \frac{-1 + \sqrt{-3}}{2} \right) &\quad \text{or} \quad x = 4 \left( \frac{-1 - \sqrt{-3}}{2} \right) \\ \Rightarrow x = 4\omega &\quad \text{or} \quad x = 4\omega^2 \end{aligned}$$

Hence cube root of  $64$  are  $4$ ,  $4\omega$  and  $4\omega^2$ .

{ Question # 2

$$\begin{aligned} (\text{i}) \quad (1 + \omega - \omega^2)^8 &= (1 + \omega + \omega^2 - 2\omega^2)^8 \\ &= (0 - 2\omega^2)^8 \quad \because 1 + \omega + \omega^2 = 0 \\ &= (-2)^8 (\omega^2)^8 = 256 \omega^{16} \\ &= 256 \omega^{15} \cdot \omega = 256 (\omega^3)^5 \cdot \omega \\ &= 256 (1)^5 \cdot \omega = 256 \omega \quad \text{Answer} \quad \because \omega^3 = 1 \end{aligned}$$

$$\begin{aligned} (\text{ii}) \quad \omega^{28} + \omega^{29} + 1 &= \omega^{27} \cdot \omega + \omega^{27} \cdot \omega^2 + 1 \\ &= (\omega^3)^9 \cdot \omega + (\omega^3)^9 \cdot \omega^2 + 1 \quad \because \omega^3 = 1 \\ &= (1)^9 \cdot \omega + (1)^9 \cdot \omega^2 + 1 \\ &= \omega + \omega^2 + 1 \\ &= 0 \quad \text{Answer} \quad \because 1 + \omega + \omega^2 = 0 \end{aligned}$$

$$\begin{aligned} (\text{iii}) \quad (1 + \omega - \omega^2)(1 - \omega + \omega^2) &= (1 + \omega + \omega^2 - 2\omega^2)(1 + \omega + \omega^2 - 2\omega) \quad \because 1 + \omega + \omega^2 = 0 \\ &= (0 - 2\omega^2)(0 - 2\omega) = (-2\omega^2)(-2\omega) \\ &= 4\omega^3 = 4(1) = 4 \quad \text{Answer} \quad \because \omega^3 = 1 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \left( \frac{-1+\sqrt{-3}}{2} \right)^7 + \left( \frac{-1-\sqrt{-3}}{2} \right)^7 \\
 &= \omega^7 + (\omega^2)^7 \\
 &= \omega^7 + \omega^{14} \\
 &= \omega^6 \cdot \omega + \omega^{12} \cdot \omega^2 = (\omega^3)^3 \cdot \omega + (\omega^3)^4 \cdot \omega^2 \\
 &= (1)^3 \cdot \omega + (1)^4 \cdot \omega^2 \\
 &= \omega + \omega^2 = -1 \quad \text{Answer} \qquad \qquad \qquad \because 1 + \omega + \omega^2 = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad & \left( -1 + \sqrt{-3} \right)^5 + \left( -1 - \sqrt{-3} \right)^5 \\
 &= \left( 2 \cdot \frac{-1 + \sqrt{-3}}{2} \right)^5 + \left( 2 \cdot \frac{-1 - \sqrt{-3}}{2} \right)^5 \\
 &= (2 \cdot \omega)^5 + (2 \cdot \omega^2)^5 \\
 &= 32\omega^5 + 32\omega^{10} = 32\omega^3 \cdot \omega^2 + 32\omega^9 \cdot \omega^1 \\
 &= 32(1) \cdot \omega^2 + 32(1) \cdot \omega \qquad \qquad \qquad \because \omega^9 = (\omega^3)^3 = (1)^3 = 1 \\
 &= 32(\omega + \omega^2) \\
 &= 32(-1) = -32 \qquad \qquad \qquad \because 1 + \omega + \omega^2 = 0
 \end{aligned}$$

{ Question # 5

$$\begin{aligned}
 \text{(i)} \quad \text{R.H.S} &= (x-y)(x-\omega y)(x-\omega^2 y) \\
 &= (x-y)[x(x-\omega^2 y) - \omega y(x-\omega^2 y)] \\
 &= (x-y)[x^2 - \omega^2 xy - \omega xy + \omega^3 y^2] \\
 &= (x-y)[x^2 - (\omega^2 + \omega)xy + (1)y^2] \qquad \qquad \qquad \because \omega^3 = 1 \\
 &= (x-y)[x^2 - (-1)xy + y^2] \qquad \qquad \qquad \because 1 + \omega + \omega^2 = 0 \\
 &= (x-y)[x^2 + xy + y^2] \qquad \qquad \qquad \because \omega + \omega^2 = -1 \\
 &= x^3 - y^3 = \text{L.H.S}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \text{R.H.S} &= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) \\
 &= (x+y+z)[x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 + \omega^2 yz \\
 &\qquad\qquad\qquad + \omega^2 xz + \omega^4 yz + \omega^3 z^2] \\
 &= (x+y+z)[x^2 + (\omega^2 + \omega)xy + (\omega + \omega^2)xz + (\omega^2 + \omega^4)yz \\
 &\qquad\qquad\qquad + \omega^3 y^2 + \omega^3 z^2] \\
 &= (x+y+z)[x^2 + (-1)xy + (-1)xz + (\omega^2 + \omega)yz + (1)y^2 + (1)z^2] \\
 &\qquad\qquad\qquad \because \omega^4 = \omega \quad \& \quad \omega + \omega^2 = -1 \\
 &= (x+y+z)[x^2 + y^2 + z^2 - xy + (-1)yz - xz] \\
 &= (x+y+z)[x^2 + y^2 + z^2 - xy - yz - xz] \\
 &= x^3 + y^3 + z^3 - 3xyz = \text{L.H.S}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \text{L.H.S} &= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8) \dots \dots \dots \text{2n factors} \\
 &= [(1+\omega)(1+\omega^2)][(1+\omega^4)(1+\omega^8)] \dots \dots \dots n \text{ factors} \\
 &= [(1+\omega)(1+\omega^2)][(1+\omega^3 \cdot \omega)(1+\omega^6 \cdot \omega^2)] \dots \dots \dots n \text{ factors} \\
 &= [(1+\omega)(1+\omega^2)][(1+\omega^3 \cdot \omega)(1+(\omega^3)^2 \cdot \omega^2)] \dots \dots \dots n \text{ factors}
 \end{aligned}$$

$$\begin{aligned}
 &= [(1 + \omega)(1 + \omega^2)][(1 + 1 \cdot \omega)(1 + (1)^2 \cdot \omega^2)] \dots \text{n factors} \\
 &= [(1 + \omega)(1 + \omega^2)][(1 + \omega)(1 + \omega^2)] \dots \text{n factors} \\
 &= [(1 + \omega)(1 + \omega^2)]^n = [1 + \omega + \omega^2 + \omega^3]^n \\
 &= [0 + 1]^n \quad \because 1 + \omega + \omega^2 = 0, \omega^3 = 1 \\
 &= [1]^n = 1 = \text{R.H.S}
 \end{aligned}$$

{ Question # 4 (i)

$$\text{Let } x^2 + x + 1 = 0 \dots (i)$$

Since  $\omega$  is root of (i) therefore

$$\omega^2 + \omega + 1 = 0 \dots (ii)$$

To prove  $\omega^2$  is root of (i)

$$\begin{aligned}
 \text{Consider } (\omega^2)^2 + \omega^2 + 1 &= \omega^4 + 2\omega^2 + 1 - \omega^2 \\
 &= (\omega^2 + 1)^2 - \omega^2 = (\omega^2 + 1 + \omega)(\omega^2 + 1 - \omega) \\
 &= (0)(\omega^2 + 1 - \omega) \quad \text{from (i)} \\
 \Rightarrow (\omega^2)^2 + \omega^2 + 1 &= 0 \dots (iii) \\
 \Rightarrow \omega^2 &\text{ is the root of the equation (i).}
 \end{aligned}$$

Now subtracting (ii) from (iii)

$$\begin{aligned}
 (\omega^2)^2 + \omega^2 + 1 &= 0 \\
 \underline{- \omega^2 - \omega - 1} &= 0 \\
 \omega^4 - \omega &= 0 \\
 \Rightarrow \omega(\omega^3 - 1) &= 0 \\
 \Rightarrow \omega^3 - 1 &= 0 \quad \text{as } \omega \neq 0 \\
 \Rightarrow \boxed{\omega^3 = 1}
 \end{aligned}$$

{ Question # 5

Let  $x$  be a cube root of  $-1$  then

$$\begin{aligned}
 x &= (-1)^{\frac{1}{3}} \Rightarrow x^3 = -1 \\
 \Rightarrow x^3 + 1 &= 0 \Rightarrow (x)^3 + (1)^3 = 0 \\
 \Rightarrow (x+1)(x^2 - x + 1) &= 0 \\
 \Rightarrow x + 1 &= 0 \quad \text{or} \quad x^2 - x + 1 = 0 \\
 \Rightarrow x &= -1 \quad \text{or} \quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\
 &= \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} \\
 \Rightarrow x &= \frac{1 + \sqrt{-3}}{2} \quad \text{or} \quad x = \frac{1 - \sqrt{-3}}{2} \\
 \Rightarrow x &= \frac{1 + \sqrt{3}i}{2} \quad \text{or} \quad x = \frac{1 - \sqrt{3}i}{2}
 \end{aligned}$$

Hence complex cube root of  $-1$  are  $\frac{1 + \sqrt{3}i}{2}$  and  $\frac{1 - \sqrt{3}i}{2}$ .

## { Question # 6

Since  $2\omega$  and  $2\omega^2$  are roots of required equation, therefore

$$\begin{aligned} & (x - 2\omega)(x - 2\omega^2) = 0 \\ \Rightarrow & x^2 - 2\omega x - 2\omega^2 x + 4\omega^3 = 0 \\ \Rightarrow & x^2 - 2x(\omega + \omega^2) + 4(1) = 0 \quad \therefore \omega^3 = 1 \\ \Rightarrow & x^2 - 2x(-1) + 4 = 0 \quad \therefore 1 + \omega + \omega^2 = 0 \\ \Rightarrow & x^2 + 2x + 4 = 0 \end{aligned}$$

is the required equation.

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## { Question # 7

(i) Let  $x$  be a fourth root of 16 then

$$\begin{aligned} x = (16)^{\frac{1}{4}} \Rightarrow x^4 &= 16 \\ \Rightarrow x^4 - 16 &= 0 \Rightarrow (x^2)^2 - (4)^2 = 0 \\ \Rightarrow (x^2 + 4)(x^2 - 4) &= 0 \\ \Rightarrow x^2 + 4 &= 0 \quad \text{or} \quad x^2 - 4 = 0 \\ \Rightarrow x^2 &= -4 \quad \text{or} \quad x^2 = 4 \\ \Rightarrow x &= \pm\sqrt{-4} \quad \text{or} \quad x = \pm\sqrt{4} \\ \Rightarrow x &= \pm 2i \quad \text{or} \quad x = \pm 2 \end{aligned}$$

Hence the four fourth root of 16 are  $2, -2, 2i, -2i$ .

(ii) Do yourself as above. **Hint:**  $81 = (9)^2$

(iii) Let  $x$  be a fourth root of 625 then

$$\begin{aligned} x = (625)^{\frac{1}{4}} \Rightarrow x^4 &= 625 \\ \Rightarrow x^4 - 625 &= 0 \Rightarrow (x^2)^2 - (25)^2 = 0 \\ \Rightarrow (x^2 + 25)(x^2 - 25) &= 0 \\ \Rightarrow x^2 + 25 &= 0 \quad \text{or} \quad x^2 - 25 = 0 \\ \Rightarrow x^2 &= -25 \quad \text{or} \quad x^2 = 25 \\ \Rightarrow x &= \pm\sqrt{-25} \quad \text{or} \quad x = \pm\sqrt{25} \\ \Rightarrow x &= \pm 5i \quad \text{or} \quad x = \pm 5 \end{aligned}$$

Hence the four fourth root of 625 are  $5, -5, 5i, -5i$ .

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**Question # 8**

$$\begin{aligned} (i) \quad 2x^4 - 32 &= 0 \\ \Rightarrow 2(x^4 - 16) &= 0 \Rightarrow x^4 - 16 = 0 \end{aligned}$$

Now do you as in Question # 7 (i)

$$\begin{aligned} (ii) \quad 3y^5 - 243y &= 0 \\ \Rightarrow 3y(y^4 - 81) &= 0 \\ \Rightarrow 3y = 0 \quad \text{or} \quad y^4 - 81 &= 0 \end{aligned}$$

$$\Rightarrow y = 0 \quad \text{or} \quad (y^2)^2 - (9)^2 = 0$$

$$\Rightarrow (y^2 + 9)(y^2 - 9) = 0$$

$$\Rightarrow y^2 + 9 = 0 \quad \text{or} \quad y^2 - 9 = 0$$

$$\Rightarrow y^2 = -9 \quad \text{or} \quad y^2 = 9$$

$$\Rightarrow y = \pm\sqrt{-9} \quad \text{or} \quad y = \pm\sqrt{9}$$

$$\Rightarrow y = \pm 3i \quad \text{or} \quad y = \pm 3$$

Hence S. Set =  $\{0, \pm 3, \pm 3i\}$

(iii)  $x^3 + x^2 + x + 1 = 0$

$$\Rightarrow x^2(x+1) + 1(x+1) = 0$$

$$\Rightarrow (x+1)(x^2 + 1) = 0$$

$$\Rightarrow x+1 = 0 \quad \text{or} \quad x^2 + 1 = 0$$

$$\Rightarrow x = -1 \quad \text{or} \quad x^2 = -1 \quad \Rightarrow x = \pm i$$

Hence S. Set =  $\{-1, \pm i\}$

(iv)  $5x^5 - 5x = 0$

$$\Rightarrow 5x(x^4 - 1) = 0$$

$$\Rightarrow 5x = 0 \quad \text{or} \quad x^4 - 1 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad (x^2)^2 - (1)^2 = 0$$

$$\Rightarrow (x^2 + 1)(x^2 - 1) = 0$$

$$\Rightarrow x^2 + 1 = 0 \quad \text{or} \quad x^2 - 1 = 0$$

$$\Rightarrow x^2 = -1 \quad \text{or} \quad x^2 = 1$$

$$\Rightarrow x = \pm i \quad \text{or} \quad x = \pm 1$$

Hence S. Set =  $\{0, \pm 1, \pm i\}$