

Chapter 4

Exercise 4.5

Remainder Theorem:-

If a polynomial $f(x)$ of degree $n \geq 1$, n is non negative integer is divided by $(x-a)$ till no x term exists in the remainder. Then $f(a)$ is a remainder.

PROOF: Suppose a polynomial $f(x)$ is divided by $(x-a)$. Then there exists a unique quotient $g(x)$ and a unique remainder R .

Dividend = Divisor \times quotient + Remainder

$$f(x) = (x-a)g(x) + R$$

Putting $x=a$ we get

$$f(a) = (a-a)g(a) + R = 0 + R = R$$

Factor Theorem:-

The polynomial $(x-a)$ is a factor of polynomial $f(x)$ if and only if $f(a)=0$ i.e. $(x-a)$ is a factor of $f(x)$ if and only if $x=a$ is a root of polynomial equation $f(x)=0$.

PROOF: Suppose $g(x)$ is the quotient and R is remainder when a polynomial $f(x)$ is divided by $(x-a)$.

Then by remainder theorem

$$f(x) = (x-a)g(x) + R$$

$$\text{Since } f(a) = 0 \Rightarrow R = 0$$

$$\therefore f(x) = (x-a)g(x)$$

$(x-a)$ is a factor of $f(x)$.

Conversely if $(x-a)$ is a factor of $f(x)$ then

$$R = f(a) = 0$$

which proves the theorem.

EXERCISE 4.5

Use the remainder theorem to find the remainder.

1. Let $f(x) = x^2 + 3x + 7$

$$\text{Sol: } x-a = x+1 \Rightarrow a = -1$$

Remainder = $f(a)$ By Remainder

$$R = f(-1) = (-1)^2 + 3(-1) + 7$$

$$= 1 - 3 + 7 = 5$$

2. Let $f(x) = x^3 - x^2 + 5x + 4$

$$x-a = x-2 \Rightarrow a = 2$$

Remainder = $f(a)$

$$R = f(2) = (2)^3 - (2)^2 + 5(2) + 4$$

$$= 8 - 4 + 10 + 4 = 18$$

3. Let $f(x) = 3x^4 + 4x^3 + x - 5$

$$x-a = x+1 \Rightarrow a = -1$$

$$R = f(a) = f(-1) = 3(-1)^4 + 4(-1)^3 - 1 - 5$$

$$= 3 - 4 - 1 - 5 = -7$$

4. $f(x) = x^3 - 2x^2 + 3x + 3$

$$x-a = x-3 \Rightarrow a = 3$$

$$R = f(a) = f(3) = 3^3 - 2(3)^2 + 3(3) + 3$$

$$= 27 - 18 + 9 + 3 = 21$$

Use factor theorem

5. $x-1, x^2 + 4x - 5$

Sol. Let $f(x) = x^2 + 4x - 5$

$$\text{and } x-a = x-1 \Rightarrow a = 1$$

Remainder = $f(a) = f(1)$

$$R = (1)^2 + 4(1) - 5 = 5 - 5 = 0$$

Hence $(x-1)$ is a factor of $f(x)$ by factor theorem.

6. Let $f(x) = x^3 + x^2 - 7x + 1$

$$x-a = x-2 \Rightarrow a = 2$$

$$R = f(a) = f(2) = 2^3 + 2^2 - 7(2) + 1$$

$$= 8 + 4 - 14 + 1 = -1 \neq 0$$

$\Rightarrow (x-2)$ is not a factor of $f(x)$

7. Let $f(w) = 2w^3 + w^2 - 4w + 7$

$$w-a = w+2 \Rightarrow a = -2$$

Remainder = $f(a) = f(-2)$

$$R = 2(-2)^3 + (-2)^2 - 4(-2) + 7$$

$$= -16 + 4 + 8 + 7 = 3 \neq 0$$

$\Rightarrow (x+2)$ is not a factor of $f(x)$

8. Let $f(n) = x^n + a^n$

where n is +ve integer

$$x-a = x-a \Rightarrow a=a$$

$$R = f(n) - f(a) = a^n - a^n = 0$$

$\Rightarrow (x-a)$ is a factor of $f(x)$

9. Let $f(x) = x^n + a^n$

where n is odd integer

$$x-a = x+a \Rightarrow a=-a$$

$$R = f(n) - f(-a) = (-a)^n + a^n$$

$$= -a^n + a^n = 0$$

$\Rightarrow (x+a)$ is a factor of $f(x)$

10. Let $f(x) = x^4 + 2x^3 + kx^2 + 3$

$$K = ? \quad R = 1$$

$$x-a = x-2 \Rightarrow a=2$$

$$R = f(2) = f(2)$$

$$1 = (2)^4 + 2(2)^3 + k(2)^2 + 3$$

$$1 = 16 + 16 + 4k + 3$$

$$\Rightarrow 4k = -34 \Rightarrow k = -\frac{17}{2}$$

11. $K = ? \quad R = 14$

Sol. $f(x) = x^3 + 2x^2 + kx + 4$

$$\frac{x-a}{x-2} = x-2 \Rightarrow a=2$$

$$14 = (2)^3 + 2(2)^2 + 2k + 4$$

$$14 = 8 + 8 + 2k + 4$$

$$\Rightarrow 2k = -6 \Rightarrow k = -3$$

Use Synthetic division...

12. Factorize the polynomial

Sol. $f(x) = x^3 + 0x^2 - 7x + 6 \quad x=2$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -7 & 6 \\ & 0 & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

$$\text{Quotient} = x^2 + 2x - 3 \quad R=0$$

$$= x^2 + 3x - x - 3$$

$$= x(x+3) - 1(x+3)$$

$$= (x+3)(x-1)$$

Hence $x^3 - 7x + 6 = (x-2)(x+3)(x+1)$

13. Let $f(x) = x^3 + 0x^2 - 28x - 48$

$$\begin{array}{r|rrrr} -4 & 1 & 0 & -28 & -48 \\ & 0 & -4 & 16 & 48 \\ \hline & 1 & -4 & -12 & 0 \end{array}$$

$$\text{Quotient} = x^2 - 4x - 12$$

$$= x^2 - 6x + 2x - 12$$

$$= x(x-6) + 2(x-6)$$

$$= (x-6)(x+2)$$

Hence

$$x^3 - 28x - 48 = (x+4)(x-6)(x+2)$$

14. Let $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & 0 & 4 & 22 & 36 & 18 \\ \hline -3 & 2 & 11 & 18 & 7 & 0 \\ & 0 & -6 & -15 & 9 & 0 \\ \hline & 2 & 5 & 3 & 0 & 0 \end{array}$$

$$\text{Quotient} = 2x^2 + 5x + 3$$

$$= 2x^2 + 2x + 3x + 3$$

$$= 2x(x+1) + 3(x+1)$$

$$= (x+1)(2x+3)$$

Hence $2x^4 + 7x^3 - 4x^2 - 27x - 18$

$$= (x-2)(x+3)(x+1)(2x+3)$$

15. Use Synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of $f(x) = x^3 + px^2 + qx + 6$

Sol. $x-a = x+1 \Rightarrow a = -1$

$$x-a = x-2 \Rightarrow a = 2$$

By Synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & p & q & 6 \\ & 0 & -1 & -p+1 & -q+p-1 \\ \hline 2 & 1 & p-1 & q-p+1 & -q+p+5 \\ & 0 & 2 & 2p+2 & \\ \hline & 1 & p+1 & q+p+3 & \end{array}$$

Since $(x+1)$ and $(x-2)$ are the factors of $f(x)$

Then Remainder = 0

$$\begin{array}{r}
 -q + p + 5 = 0 \quad \dots \dots (1) \\
 q + p + 3 = 0 \quad \dots \dots (2) \\
 \hline
 2p + 8 = 0 \Rightarrow p = -4
 \end{array}$$

put $p = -4$ in (2)

$$\begin{array}{l}
 \Rightarrow q - 4 + 3 = 0 \Rightarrow q = 1 \\
 p = -4 \quad q = 1
 \end{array}$$

16. find the values of a and b if -2 and 2 are the roots of $f(x) = x^3 - 4x^2 + ax + b$

Sol. $a = -2 \quad b = 2$

-2	1	-4	a	b
	0	-2	12	$-2a - 24$
2	1	-6	$a + 12$	$b - 2a - 24$
	0	2	-8	

$$1 - 4 + a + 4$$

Since $-2, 2$ are the roots of $f(x)$ then Remainder $= 0$

$$a + 4 = 0 \Rightarrow a = -4$$

$$b - 2a - 24 = 0$$

$$\Rightarrow b - 2(-4) - 24 = 0$$

$$\Rightarrow b + 8 - 24 = 0$$

$$\Rightarrow b - 16 = 0 \Rightarrow b = 16$$

$$a = -4 \quad b = 16$$

Relation Between Roots &

Coefficients of Quadratic Eq

Quadratic eq, $ax^2 + bx + c = 0$

$$\text{formula } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

let α, β be the roots of the eq

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a} = \text{sum of roots}$$

$$\alpha \beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right)$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{2a^2}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Product of roots $= \frac{c}{a}$

Formation of an Eq

whose Roots are given

If α, β be the roots of the quadratic equation

$$\text{then } x = \alpha \quad x = \beta$$

$$\Rightarrow x - \alpha = 0 \quad x - \beta = 0$$

Equation becomes

$$(x - \alpha)(x - \beta) = 0$$

$$\Rightarrow x^2 - \beta x - \alpha x + \alpha \beta = 0$$

$$\Rightarrow x^2 - (\alpha + \beta)x + \alpha \beta = 0$$

$$\Rightarrow x^2 - Sx + P = 0$$