

# Chapter 5

## Exercise 5.1

Resolving the following into partial fractions:

## **Question # 1**

$$\frac{1}{x^2 - 1}$$

### **Solution**

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)}$$

Now suppose

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Multiplying both sides by  $(x-1)(x+1)$  we get

$$1 = A(x+1) + B(x-1) \quad \dots \dots \dots \text{(i)}$$

Put  $x-1=0 \Rightarrow x=1$  in equation (i)

$$1 = A(1+1) + B(0) \Rightarrow 1 = 2A + 0 \Rightarrow A = \frac{1}{2}$$

Now put  $x+1=0 \Rightarrow x=-1$  in equation (i)

$$1 = A(0) + B(-1 - 1) \Rightarrow 1 = 0 - 2B \Rightarrow B = -\frac{1}{2}$$

Hence

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

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## Question # 2

$$\frac{x^2+1}{(x+1)(x-1)}$$

$$\begin{array}{r} \frac{1}{x^2+1} \\ - \quad + \\ \hline 2 \end{array}$$

### **Solution**

$$\frac{x^2+1}{(x+1)(x-1)} = \frac{x^2+1}{x^2-1}$$

$$= 2 + \frac{1}{x^2-1} = 2 + \frac{1}{(x+1)(x-1)}$$

Now consider

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

Multiplying both sides by  $(x+1)(x-1)$

$$2 = A(x-1) + B(x+1) \quad \dots \dots \dots \text{(i)}$$

Put  $x+1=0 \Rightarrow x=-1$  in equation (i)

$$2 = A(-1-1) + B(0) \Rightarrow 2 = -2A + 0 \Rightarrow A = -1$$

Now put  $x-1=0 \Rightarrow x=1$  in equation (i)

$$2 = A(0) + B(1+1) \Rightarrow 2 = 0 + 2B \Rightarrow \boxed{B=1}$$

$$\text{So } \frac{2}{(x+1)(x-1)} = \frac{-1}{x+1} + \frac{1}{x-1}$$

Hence

$$\begin{aligned}\frac{x^2+1}{(x+1)(x-1)} &= 2 + \frac{-1}{(x+1)} + \frac{1}{(x-1)} \\ &= 2 - \frac{1}{(x+1)} + \frac{1}{(x-1)}\end{aligned}\quad \text{Answer}$$

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**Question # 3**

$$\frac{2x+1}{(x-1)(x+2)(x+3)}$$

### **Solution**

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Multiplying both side by  $(x-1)(x+2)(x+3)$

Put  $x-1=0 \Rightarrow x=1$  in equation (i)

$$2(1) + 1 = A(1+2)(1+3) + B(0) + C(0)$$

$$3 = A(3)(4) + 0 + 0 \quad \Rightarrow \quad 3 = 12A \quad \Rightarrow \quad \frac{3}{12} = A \quad \Rightarrow \quad A = \frac{1}{4}$$

Now put  $x + 2 = 0 \Rightarrow x = -2$  in equation (i)

$$2(-2) + 1 = A(0) + B(-2 - 1)(-2 + 3) + C(0)$$

$$-4 + 1 = 0 + B(-3)(1) + 0 \Rightarrow -3 = -3B \Rightarrow B = 1$$

Now put  $x + 3 = 0 \Rightarrow x = -3$  in equation (i)

$$2(-3) + 1 = A(0) + B(0) + C(-3 - 1)(-3 + 2)$$

$$-6+1 = 0+0+C(-4)(-1) \Rightarrow -5 = 4C \Rightarrow C = -\frac{5}{4}$$

S<sub>0</sub>

$$\begin{aligned}\frac{2x+1}{(x-1)(x+2)(x+3)} &= \frac{\cancel{1}/4}{x-1} + \frac{1}{x+2} + \frac{-\cancel{5}/4}{x+3} \\ &= \frac{1}{4(x-1)} + \frac{1}{x+2} - \frac{5}{4(x+3)}\end{aligned}\quad \text{Answer}$$

**Question # 4**

$$\frac{3x^2 - 4x - 5}{(x-2)(x^2 + 7x + 10)} \quad \because x^2 + 7x + 10 = x^2 + 5x + 2x + 10 \\ = x(x+5) + 2(x+5)$$

**Solution**

$$\frac{3x^2 - 4x - 5}{(x-2)(x^2 + 7x + 10)} = \frac{3x^2 - 4x - 5}{(x-2)(x+5)(x+2)} \\ = (x+5)(x+2)$$

Now resolving into partial fraction.

$$\frac{3x^2 - 4x - 5}{(x-2)(x+5)(x+2)} = \frac{A}{x-2} + \frac{B}{x+5} + \frac{C}{x+2}$$

*Do yourself. You will get*

$$\left[ \begin{array}{l} A = -\frac{1}{28}, B = \frac{30}{7}, C = -\frac{5}{4} \end{array} \right]$$

**Question # 5**

$$\frac{1}{(x-1)(2x-1)(3x-1)}$$

**Solution**

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1} + \frac{B}{2x-1} + \frac{C}{3x-1}$$

Multiplying both side by  $(x-1)(2x-1)(3x-1)$ .

$$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(2x-1)(3x-1) \dots \dots \dots \text{(i)}$$

Put  $x-1=0 \Rightarrow x=1$  in equation (i)

$$1 = A(2(1)-1)(3(1)-1) + B(0) + C(0) \Rightarrow 1 = A(1)(2) + 0 + 0$$

$$\Rightarrow 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$$

Put  $2x-1=0 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$  in equation (i)

$$1 = A(0) + B\left(\frac{1}{2}-1\right)\left(3\left(\frac{1}{2}\right)-1\right) + C(0) \Rightarrow 1 = 0 + B\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + 0$$

$$\Rightarrow 1 = -\frac{1}{4}B \Rightarrow \boxed{B = -4}$$

Put  $3x-1=0 \Rightarrow 3x=1 \Rightarrow x=\frac{1}{3}$  in equation (i)

$$1 = A(0) + B(0) + C\left(\frac{1}{3}-1\right)\left(2\left(\frac{1}{3}\right)-1\right) \Rightarrow 1 = 0 + 0 + C\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right)$$

$$\Rightarrow 1 = \frac{2}{9}C \Rightarrow \boxed{C = \frac{9}{2}}$$

Hence  $\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{1/2}{x-1} + \frac{-4}{2x-1} + \frac{9/2}{3x-1}$   
 $= \frac{1}{2(x-1)} - \frac{4}{2x-1} + \frac{9}{2(3x-1)}$  Answer

**Question # 6**

$$\frac{x}{(x-a)(x-b)(x-c)}$$

**Solution**

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

Multiplying both sides by  $(x-a)(x-b)(x-c)$ .

$$x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \dots \dots \dots \text{(i)}$$

Put  $x-a=0 \Rightarrow x=a$  in equation (i)

$$a = A(a-b)(a-c) + B(0) + C(0)$$

$$\Rightarrow a = A(a-b)(a-c) + 0 + 0 \Rightarrow A = \boxed{\frac{a}{(a-b)(a-c)}}$$

Now put  $x-b=0 \Rightarrow x=b$  in equation (i)

$$a = A(0) + B(b-a)(b-c) + C(0)$$

$$\Rightarrow a = 0 + B(b-a)(b-c) + 0 \Rightarrow B = \boxed{\frac{b}{(b-a)(b-c)}} \text{ Now put}$$

$x-c=0 \Rightarrow x=c$  in equation (i)

$$c = A(0) + B(0) + C(c-a)(c-b)$$

$$\Rightarrow c = 0 + 0 + C(c-a)(c-b) \Rightarrow C = \boxed{\frac{c}{(c-a)(c-b)}}$$

So

$$\begin{aligned} \frac{x}{(x-a)(x-b)(x-c)} &= \frac{a/(a-b)(a-c)}{x-a} + \frac{b/(b-a)(b-c)}{x-b} + \frac{c/(c-a)(c-b)}{x-c} \\ &= \frac{a}{(a-b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-a)(c-b)(x-c)} \end{aligned}$$

*Answer*

**Question # 7**

$$\frac{6x^3+5x^2-7}{2x^2-x-1}$$

**Solution**

$$\frac{6x^3+5x^2-7}{2x^2-x-1}$$

$$\begin{aligned}
 &= 3x + 4 + \frac{7x - 3}{2x^2 - x - 1} \\
 &= 3x + 4 + \frac{7x - 3}{2x^2 - 2x + x - 1} \\
 &= 3x + 4 + \frac{7x - 3}{2x(x-1) + 1(x-1)} \\
 &= 3x + 4 + \frac{7x - 3}{(x-1)(2x+1)}
 \end{aligned}$$

$$\begin{array}{r} & 3x + 4 \\ \hline 2x^2 - x - 1 & | 6x^3 + 5x^2 - 7 \\ & \underline{-} 6x^3 - 3x^2 - 3x \\ & \quad + \quad + \\ & 8x^2 + 3x - 7 \\ & \underline{-} 8x^2 - 4x - 4 \\ & \quad + \quad + \\ & 7x - 3 \end{array}$$

### Now Consider

$$\frac{7x-3}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$$

*Find value of A & B yourself*  
*You will get A =  $\frac{4}{3}$  and B =  $\frac{13}{3}$*

so

$$\frac{7x-3}{(x-1)(2x+1)} = \frac{\cancel{4}/_3}{x-1} + \frac{\cancel{13}/_3}{2x+1} = \frac{4}{3(x-1)} + \frac{13}{3(2x+1)}$$

Hence

$$\frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1} = 3x + 4 + \frac{4}{3(x-1)} + \frac{13}{3(2x+1)} \quad \text{Answer}$$

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## **Question # 8**

$$\frac{2x^3 + x^2 - 5x + 3}{2x^3 + x^2 - 3x}$$

$$\begin{array}{r} & & 1 \\ & 2x^3 + x^2 - 3x \sqrt{2x^3 + x^2 - 5x + 3} \\ & \underline{-} & 2x^3 + x^2 - 3x \\ & & - & + \\ & & & 2x + 3 \end{array}$$

### **Solution**

$$\begin{aligned}
 & \frac{2x^3 + x^2 - 5x + 3}{2x^3 + x^2 - 3x} \\
 = 1 + & \frac{-2x + 3}{2x^3 + x^2 - 3x} \\
 = 1 + & \frac{-2x + 3}{x(2x^2 + x - 3)} \\
 = 1 + & \frac{-2x - 3}{x(x(2x + 3) - 1)}
 \end{aligned}$$

$$= 1 + \frac{-2x+3}{x(2x^2+3x-2x-3)}$$

Now consider

$$\frac{3-2x}{x(2x+3)(x-1)} = \frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x-1}$$

Put  $x=0$  in equation (i)

$$3 - 2(0) = A(2(0) + 3)((0) - 1) + B(0) + C(0) \Rightarrow 3 - 0 = A(0 + 3)(-1) + 0 + 0$$

$$\Rightarrow 3 = -3A \quad \Rightarrow \boxed{A = -1}$$

Now put  $2x+3=0 \Rightarrow 2x=-3 \Rightarrow x=-\frac{3}{2}$  in equation (i)

$$\begin{aligned} 3-2\left(-\frac{3}{2}\right) &= A(0)+B\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)+C(0) \Rightarrow 3+3=0+B\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)+0 \\ \Rightarrow 6 &= \frac{15}{4}B \Rightarrow B=\left(6\right)\left(\frac{4}{15}\right) \Rightarrow \boxed{B=\frac{8}{5}} \end{aligned}$$

Now put  $x-1=0 \Rightarrow x=1$  in equation (i)

$$3-2(1)=A(0)+B(0)+C(1)(2(1)+3) \Rightarrow 1=0+0+5C \Rightarrow \boxed{C=\frac{1}{5}}$$

$$\text{So } \frac{3-2x}{x(2x+3)(x-1)} = \frac{-1}{x} + \frac{\cancel{8}/5}{2x+3} + \frac{\cancel{1}/5}{x-1} = -\frac{1}{x} + \frac{8}{5(2x+3)} + \frac{1}{5(x-1)}$$

$$\text{Hence } \frac{2x^3+x^2-5x+3}{2x^3+x^2-3x} = 1 - \frac{1}{x} + \frac{8}{5(2x+3)} + \frac{1}{5(x-1)} \quad \text{Answer}$$

**Question # 9**

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$$

**Solution**

$$\begin{aligned} \frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} &= \frac{(x-1)(x^2-3x-5x+15)}{(x-2)(x^2-4x-6x+24)} \\ &= \frac{(x-1)(x^2-8x+15)}{(x-2)(x^2-10x+24)} = \frac{x^3-8x^2+15x-x^2+8x-15}{x^3-10x^2+24x-2x^2+20x-48} \\ &= \frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48} \\ &= 1 + \frac{3x^2-21x+33}{x^3-12x^2+44x-48} \quad x^3-12x^2+44x-48 \overline{)x^3-9x^2+23x-15} \\ &= 1 + \frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} \quad \frac{x^3-12x^2+44x-48}{x^3-12x^2+44x-48} \\ &\qquad\qquad\qquad \underline{-} \quad \underline{+} \quad \underline{-} \quad \underline{+} \\ &\qquad\qquad\qquad \underline{3x^2-21x+33} \end{aligned}$$

Now Suppose

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{A}{x-2} + \frac{B}{x-4} + \frac{C}{x-6}$$

$\left[ \begin{array}{l} \text{Find value of } A, B \text{ and } C \text{ yourself} \\ \text{You will get } A=\cancel{3}/8, B=\cancel{3}/4, C=\cancel{15}/8 \end{array} \right]$

$$\begin{aligned} \text{So } \frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} &= \frac{\cancel{3}/8}{x-2} + \frac{\cancel{3}/4}{x-4} + \frac{\cancel{15}/8}{x-6} \\ &= \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)} \end{aligned}$$

Hence

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = 1 + \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)} \quad \text{Answer}$$

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**Question # 10**

$$\frac{1}{(1-ax)(1-bx)(1-cx)}$$

### *Solution*

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A}{1-ax} + \frac{B}{1-bx} + \frac{C}{1-cx}$$

Multiplying both sides by  $(1 - ax)(1 - bx)(1 - cx)$ .

Put  $1-ax=0 \Rightarrow ax=1 \Rightarrow x=\frac{1}{a}$  in equation (i).

$$1 = A \left( 1 - b \cdot \frac{1}{a} \right) \left( 1 - c \cdot \frac{1}{a} \right) + B(0) + C(0) \quad \Rightarrow 1 = A \left( 1 - \frac{b}{a} \right) \left( 1 - \frac{c}{a} \right) + 0 + 0$$

$$1 = A \left( \frac{a-b}{a} \right) \left( \frac{a-c}{a} \right) \quad \Rightarrow 1 = A \frac{(a-b)(a-c)}{a^2} \quad \Rightarrow \boxed{A = \frac{a^2}{(a-b)(a-c)}}$$

Find value of  $B$  &  $C$  yourself as  $A$ .  
 You will get  $B = \frac{b^2}{(b-a)(b-c)}$ ,  $C = \frac{c^2}{(c-a)(c-b)}$

$$\text{Hence } \frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{\frac{a^2}{(a-b)(a-c)}}{1-ax} + \frac{\frac{b^2}{(b-a)(b-c)}}{1-bx} + \frac{\frac{c^2}{(c-a)(c-b)}}{1-cx}$$

$$= \frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-a)(b-c)(1-bx)} + \frac{c^2}{(c-a)(c-b)(1-cx)}$$

### *Answer*

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**Question # 11**

$$\frac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)}$$

### *Solution*

$$\frac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)}$$

Put  $y = x^2$  in above.

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)}$$

Now consider

Put  $y + b^2 = 0 \Rightarrow y = -b^2$  in equation (i)

$$-b^2 + a^2 = A(-b^2 + c^2)(-b^2 + d^2) + B(0) + C(0)$$

$$\Rightarrow a^2 - b^2 = A(c^2 - b^2)(d^2 - b^2) + 0 + 0 \Rightarrow A = \boxed{\frac{a^2 - b^2}{(c^2 - b^2)(d^2 - b^2)}}$$

Now put  $y + c^2 = 0 \Rightarrow y = -c^2$  in equation (i)

$$-c^2 + a^2 = A(0) + B(-c^2 + b^2)(-b^2 + d^2) + C(0)$$

$$\Rightarrow a^2 - c^2 = 0 + B(b^2 - c^2)(d^2 - c^2) + 0 \Rightarrow B = \boxed{\frac{a^2 - c^2}{(b^2 - c^2)(d^2 - c^2)}}$$

Now put  $y + d^2 = 0 \Rightarrow y = -d^2$  in equation (i)

$$-d^2 + a^2 = A(0) + B(0) + C(-d^2 + b^2)(-d^2 + c^2)$$

$$\Rightarrow a^2 - d^2 = 0 + 0 + C(b^2 - d^2)(c^2 - d^2) \Rightarrow C = \boxed{\frac{a^2 - d^2}{(b^2 - d^2)(c^2 - d^2)}}$$

Hence

$$\begin{aligned} \frac{y + a^2}{(y + b^2)(y + c^2)(y + d^2)} &= \frac{a^2 - b^2}{(c^2 - b^2)(d^2 - b^2)} + \frac{a^2 - c^2}{(b^2 - c^2)(d^2 - c^2)} + \frac{a^2 - d^2}{(b^2 - d^2)(c^2 - d^2)} \\ &= \frac{a^2 - b^2}{(c^2 - b^2)(d^2 - b^2)(y + b^2)} + \frac{a^2 - c^2}{(b^2 - c^2)(d^2 - c^2)(y + c^2)} + \frac{a^2 - d^2}{(b^2 - d^2)(c^2 - d^2)(y + d^2)} \end{aligned}$$

Since  $y = x^2$

$$= \frac{a^2 - b^2}{(c^2 - b^2)(d^2 - b^2)(x^2 + b^2)} + \frac{a^2 - c^2}{(b^2 - c^2)(d^2 - c^2)(x^2 + c^2)} + \frac{a^2 - d^2}{(b^2 - d^2)(c^2 - d^2)(x^2 + d^2)}$$


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