

Chapter 5

Exercise 5.3

Question # 1

$$\frac{9x-7}{(x^2+1)(x+3)}$$

Solution

$$\frac{9x-7}{(x^2+1)(x+3)}$$

Resolving it into partial fraction.

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+3)}$$

Multiplying both sides by $(x^2+1)(x+3)$.

$$9x-7 = (Ax+B)(x+3) + C(x^2+1) \dots (i)$$

Put $x+3=0 \Rightarrow x=-3$ in equation (i).

$$9(-3)-7 = (A(-3)+B)(0) + C((-3)^2+1) \Rightarrow -27-7 = 0 + C(9+1)$$
$$\Rightarrow -34 = 10C \Rightarrow C = -\frac{34}{10} \Rightarrow C = -\frac{17}{5}$$

Now equation (i) can be written as

$$9x-7 = A(x^2+3x) + B(x+3) + C(x^2+1)$$

Comparing the coefficients of x^2 , x and x^0 .

$$0 = A + C \dots (ii)$$

$$9 = 3A + B \dots (iii)$$

$$-7 = +3B + C \dots (iv)$$

Putting value of C in equation (ii)

$$0 = A - \frac{17}{5} \Rightarrow A = \frac{17}{5}$$

Now putting value of A in equation (iii)

$$9 = 3\left(\frac{17}{5}\right) + B \Rightarrow 9 = \frac{51}{5} + B$$

$$\Rightarrow 9 - \frac{51}{5} = B \Rightarrow B = -\frac{6}{5}$$

Hence

$$\begin{aligned} \frac{9x-7}{(x^2+1)(x+3)} &= \frac{\frac{17}{5}x - \frac{6}{5}}{x^2+1} + \frac{-\frac{17}{5}}{(x+3)} \\ &= \frac{\frac{17x-6}{5}}{x^2+1} - \frac{\frac{17}{5}}{(x+3)} = \frac{17x-6}{5(x^2+1)} - \frac{17}{5(x+3)} \end{aligned} \quad \text{Answer}$$

Question # 2 $\frac{1}{(x^2+1)(x+1)}$

Solution $\frac{1}{(x^2+1)(x+1)}$

Now Consider

$$\frac{1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

Multiplying both sides by $(x^2+1)(x+1)$.

$$1 = (Ax+B)(x+1) + C(x^2+1) \dots\dots\dots (i)$$

Put $x+1=0 \Rightarrow x=-1$ in equation (i)

$$1 = 0 + C((-1)^2+1) \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$$

Now eq. (i) can be written as

$$1 = A(x^2+x) + B(x+1) + C(x^2+1)$$

Comparing the coefficients of x^2 , x and x^0 .

$$0 = A + C \dots (ii)$$

$$0 = A + B \dots (iii)$$

$$1 = A + C \dots (iv)$$

Putting value of C in equation (ii)

$$0 = A + \frac{1}{2} \Rightarrow A = -\frac{1}{2}$$

Putting value of A in equation (iii)

$$0 = -\frac{1}{2} + B \Rightarrow B = \frac{1}{2}$$

$$\begin{aligned} \text{Hence } \frac{1}{(x^2+1)(x+1)} &= \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x+1} = \frac{-x+1}{2(x^2+1)} + \frac{1}{2(x+1)} \\ &= \frac{-x+1}{2(x^2+1)} + \frac{1}{2(x+1)} = \frac{1-x}{2(x^2+1)} + \frac{1}{2(x+1)} \end{aligned} \quad \text{Answer}$$

Question # 3 $\frac{3x+7}{(x^2+4)(x+3)}$

Solution $\frac{3x+7}{(x^2+4)(x+3)}$

Resolving it into partial fraction.

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+3}$$

Now do yourself, you will get
 $A = \frac{2}{13}, B = \frac{33}{13} \text{ and } C = -\frac{2}{13}$

Question # 4 $\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)}$

Solution $\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)}$

Resolving it into partial fraction.

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{Ax + B}{x^2 + 2x + 5} + \frac{C}{x - 1}$$

$$\Rightarrow x^2 + 15 = (Ax + B)(x - 1) + C(x^2 + 2x + 5) \dots (\text{i})$$

Put $x - 1 = 0 \Rightarrow x = 1$ in equation (i)

$$(1)^2 + 15 = (A(1) + B)(0) + C((1)^2 + 2(1) + 5)$$

$$\Rightarrow 1 + 15 = 0 + C(1 + 2 + 5)$$

$$\Rightarrow 16 = 8C \Rightarrow \frac{16}{8} = C \Rightarrow C = 2$$

Now equation (i) can be written as

$$x^2 + 15 = A(x^2 - x) + B(x - 1) + C(x^2 + 2x + 5)$$

Comparing the coefficients of x^2 , x and x^0 .

$$1 = A + C \dots (\text{ii})$$

$$0 = -A + B + 2C \dots (\text{iii})$$

$$15 = -B + 5C \dots (\text{iv})$$

Putting value of C in equation (ii).

$$1 = A + 2 \Rightarrow 1 - 2 = A$$

$$\Rightarrow A = -1$$

Putting value of A and C in equation (iii)

$$0 = -(-1) + B + 2(2) \Rightarrow 0 = 1 + B + 4$$

$$\Rightarrow 0 = B + 5 \Rightarrow B = -5$$

Hence

$$\begin{aligned} \frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} &= \frac{(-1)x - 5}{x^2 + 2x + 5} + \frac{2}{x - 1} \\ &= \frac{-x - 5}{x^2 + 2x + 5} + \frac{2}{x - 1} \quad \text{Answer} \end{aligned}$$

Question # 5 $\frac{x^2}{(x^2 + 4)(x + 2)}$

Solution $\frac{x^2}{(x^2 + 4)(x + 2)}$

Resolving it into partial fraction.

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+2}$$

Now do yourself, you will get

$$A = \frac{1}{2}, B = -1 \text{ and } C = -\frac{1}{2}$$

Question # 6 $\frac{x^2+1}{x^3+1}$

Solution
$$\frac{x^2+1}{x^3+1} = \frac{x^2+1}{(x+1)(x^2-x+1)} \quad \because x^3+1=(x+1)(x^2-x+1)$$

Now consider

$$\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad \dots (\text{A})$$

$$x^2+1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$x^2+1 = A(x^2-x+1) + (Bx^2+Bx+Cx+C)$$

$$x^2+1 = (A+B)x^2 + (B+C-A)x + (A+C)$$

By comparing the coefficients of x^2 , x and x^0

$$A+B=1 \quad \dots (1)$$

$$B+C-A=0 \quad \dots (2)$$

$$A+C=1 \quad \dots (3)$$

From (3), We have

$$C=1-A$$

Put the value of C in (2).

$$-2A+B=-1 \quad \dots (4)$$

Subtract (1) from (4), We have

$$3A=2 \Rightarrow A=\frac{2}{3}$$

Put the value of A in (1) and (3), we have

$$B=\frac{1}{3}, \quad C=\frac{1}{3}$$

Put the values of A , B and C in (A), We have

$$\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{2}{3(x+1)} + \frac{(x+1)}{3(x^2-x+1)}$$

Question # 7 $\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$

Solution
$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$$

Consider

$$\frac{x^2 + 2x + 2}{(x^2 + 3)(x+1)(x-1)} = \frac{Ax+B}{x^2+3} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$\Rightarrow x^2 + 2x + 2 = (Ax+B)(x+1)(x-1) + C(x^2+3)(x-1) + D(x^2+3)(x+1) \dots (i)$$

Put $x+1=0 \Rightarrow x=-1$ in equation (i)

$$(-1)^2 + 2(-1) + 2 = 0 + C((-1)^2 + 3)((-1)-1) + 0 \Rightarrow 1 - 2 + 2 = C(4)(-2)$$

$$\Rightarrow 1 = -8C \Rightarrow C = -\frac{1}{8}$$

Now put $x-1=0 \Rightarrow x=1$ in equation (i)

$$\Rightarrow (1)^2 + 2(1) + 2 = 0 + 0 + D((1)^2 + 3)((1)+1) \Rightarrow 1 + 2 + 2 = D(4)(2)$$

$$\Rightarrow 5 = 8D \Rightarrow D = \frac{5}{8}$$

Equation (i) can be written as

$$x^2 + 2x + 2 = (Ax+B)(x^2-1) + C(x^3-x^2+3x-3) + D(x^3+x^2+3x+3)$$

$$\Rightarrow x^2 + 2x + 2 = A(x^3-x) + B(x^2-1) + C(x^3-x^2+3x-3) + D(x^3+x^2+3x+3)$$

Comparing the coefficients of x^3 , x^2 , x and x^0 .

$$0 = A + C + D \dots (ii)$$

$$1 = B - C + D \dots (iii)$$

$$2 = -A + 3C + 3D \dots (iv)$$

$$2 = -B - 3C + 3D \dots (v)$$

Putting values of C and D in (ii)

$$0 = A - \frac{1}{8} + \frac{5}{8} \Rightarrow 0 = A + \frac{1}{2} \Rightarrow A = -\frac{1}{2}$$

Putting values of C and D in (iii)

$$1 = B - \left(-\frac{1}{8}\right) + \frac{5}{8} \Rightarrow 1 = B + \frac{1}{8} + \frac{5}{8} \Rightarrow 1 = B + \frac{3}{4}$$

$$\Rightarrow 1 - \frac{3}{4} = B \Rightarrow B = \frac{1}{4}$$

Hence

$$\begin{aligned} \frac{x^2 + 2x + 2}{(x^2 + 3)(x+1)(x-1)} &= \frac{-\frac{1}{2}x + \frac{1}{4}}{x^2+3} + \frac{-\frac{1}{8}}{x+1} + \frac{\frac{5}{8}}{x-1} \\ &= \frac{-2x+1}{4(x^2+3)} + \frac{-1}{8(x+1)} + \frac{5}{8(x-1)} \end{aligned}$$

$$= \frac{1-2x}{4(x^2+3)} - \frac{1}{8(x+1)} + \frac{5}{8(x-1)} \quad \text{Answer}$$

Question # 8 $\frac{1}{(x-1)^2(x^2+2)}$

Solution $\frac{1}{(x-1)^2(x^2+2)}$

Resolving it into partial fraction.

$$\frac{1}{(x-1)^2(x^2+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2}$$

$$\Rightarrow 1 = A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2 \dots (\text{i})$$

Put $x-1=0 \Rightarrow x=1$ in equation (i)

$$1 = 0 + B((1)^2 + 2) + 0$$

$$\Rightarrow 1 = 3B \Rightarrow \boxed{B = \frac{1}{3}}$$

Now equation (i) can be written as

$$1 = A(x^3 - x^2 + 2x - 2) + B(x^2 + 2) + (Cx + D)(x^2 - 2x + 1)$$

$$\Rightarrow 1 = A(x^3 - x^2 + 2x - 2) + B(x^2 + 2) + C(x^3 - 2x^2 + x) + D(x^2 - 2x + 1)$$

Comparing the coefficients of x^3 , x^2 , x and x^0 .

$$0 = A + C \dots (\text{ii})$$

$$0 = -A + B - 2C + D \dots (\text{iii})$$

$$0 = 2A + C - 2D \dots (\text{iv})$$

$$1 = -2A + 2B + D \dots (\text{v})$$

Multiplying eq. (iii) by 2 and adding in (iv)

$$\begin{aligned} 0 &= -2A + 2B - 4C + 2D \\ 0 &= 2A + C - 2D \\ \hline 0 &= 2B - 3C \end{aligned}$$

Putting value of B in above

$$0 = 2\left(\frac{1}{3}\right) - 3C \Rightarrow 0 = \frac{2}{3} - 3C \Rightarrow 3C = \frac{2}{3} \Rightarrow \boxed{C = \frac{2}{9}}$$

Putting value of C in eq. (ii)

$$0 = A + \frac{2}{9} \Rightarrow \boxed{A = -\frac{2}{9}}$$

Putting value of A and B in eq. (v)

$$1 = -2\left(-\frac{2}{9}\right) + 2\left(\frac{1}{3}\right) + D \Rightarrow 1 = \frac{4}{9} + \frac{2}{3} + D$$

$$\Rightarrow 1 - \frac{4}{9} - \frac{2}{3} = D \Rightarrow \boxed{D = -\frac{1}{9}}$$

Hence

$$\begin{aligned}\frac{1}{(x-1)^2(x^2+2)} &= \frac{-\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{\left(\frac{2}{9}\right)x + \left(-\frac{1}{9}\right)}{x^2+2} \\ &= \frac{-\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{\frac{2x-1}{9}}{x^2+2} \\ &= \frac{-2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{2x-1}{9(x^2+2)}\end{aligned}$$

Question # 9 $\frac{x^4}{1-x^4}$

Solution $\frac{x^4}{1-x^4}$

$$\begin{aligned}&= -1 + \frac{1}{1-x^4} = -1 + \frac{1}{(1-x^2)(1+x^2)} \\ &= -1 + \frac{1}{(1-x)(1+x)(1+x^2)}\end{aligned}$$

$$\begin{array}{r} -1 \\ 1 - x^4 \overline{)x^4} \\ \underline{-x^4 - 1} \\ \hline 1 \end{array}$$

Now consider

$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

Now find values of A, B, C and D yourself.
You will get A = $\frac{1}{4}$, B = $\frac{1}{4}$, C = 0 and D = $\frac{1}{2}$

So

$$\begin{aligned}\frac{1}{(1-x)(1+x)(1+x^2)} &= \frac{\frac{1}{4}}{1-x} + \frac{\frac{1}{4}}{1+x} + \frac{(0)x + \frac{1}{2}}{1+x^2} \\ &= \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}\end{aligned}$$

Hence

$$\frac{x^4}{1-x^4} = -1 + \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)} \quad \text{Answer}$$

Question # 10 $\frac{x^2-2x+3}{x^4+x^2+1}$

Solution $\frac{x^2-2x+3}{x^4+x^2+1}$

$$= \frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}$$

$$\begin{aligned}\because x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 \\ &= (x^2+1)^2 - x^2 \\ &= (x^2+1+x)(x^2+1-x) \\ &= (x^2+x+1)(x^2-x+1)\end{aligned}$$

Now Consider

$$\begin{aligned} \frac{x^2 - 2x + 3}{(x^2 + x + 1)(x^2 - x + 1)} &= \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 - x + 1} \\ \Rightarrow x^2 - 2x + 3 &= (Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1) \dots (\text{i}) \\ \Rightarrow x^2 - 2x + 3 &= A(x^3 - x^2 + x) + B(x^2 - x + 1) + C(x^3 + x^2 + x) + D(x^2 + x + 1) \end{aligned}$$

Comparing the coefficients of x^3 , x^2 , x and x^0 .

$$\begin{aligned} 0 &= A + C \dots (\text{ii}) \\ 1 &= -A + B + C + D \dots (\text{iii}) \\ -2 &= A - B + C + D \dots (\text{iv}) \\ 3 &= B + D \dots (\text{v}) \end{aligned}$$

Subtracting (ii) and (iv)

$$\begin{array}{r} 0 = A + C \\ -2 = A - B + C + D \\ \hline 2 = B - D \end{array}$$

$$\Rightarrow 2 = B - D \dots (\text{vi})$$

Adding (v) and (vi)

$$\begin{array}{r} 3 = B + D \\ 2 = B - D \\ \hline 5 = 2B \\ \Rightarrow B = \frac{5}{2} \end{array}$$

Putting value of B in (v)

$$\begin{array}{l} 3 = \frac{5}{2} + D \\ \Rightarrow 3 - \frac{5}{2} = D \quad \Rightarrow D = \frac{1}{2} \end{array}$$

Putting value of B and D in (iii)

$$\begin{array}{l} 1 = -A + \frac{5}{2} + C + \frac{1}{2} \\ \Rightarrow 1 - \frac{5}{2} - \frac{1}{2} = -A + C \\ \Rightarrow -2 = -A + C \dots (\text{vii}) \end{array}$$

Adding (ii) and (vii)

$$\begin{array}{r} 0 = A + C \\ -2 = -A + C \\ \hline -2 = 2C \\ \Rightarrow C = -1 \end{array}$$

Putting value of C in equation (ii)

$$\begin{array}{l} 0 = A - 1 \\ \Rightarrow A = 1 \end{array}$$

Hence

$$\begin{aligned}\frac{x^2 - 2x + 3}{(x^2 + x + 1)(x^2 - x + 1)} &= \frac{(1)x + \frac{5}{2}}{x^2 + x + 1} + \frac{(-1)x + \frac{1}{2}}{x^2 - x + 1} \\ &= \frac{\underline{2x + 5}}{x^2 + x + 1} + \frac{\underline{-2x + 1}}{x^2 - x + 1} \\ &= \frac{2x + 5}{2(x^2 + x + 1)} + \frac{-2x + 1}{2(x^2 - x + 1)} \\ &= \frac{2x + 5}{2(x^2 + x + 1)} + \frac{1 - 2x}{2(x^2 - x + 1)} \quad \text{Answer}\end{aligned}$$
